which follow from the alternative definitions in (4.37) and are directly related by the total antisymmetry of $\gamma^0 G_+(x-x')$. Because the Dirac Green's function satisfies

$$\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right) G_+(x-x') = (m^2 - \partial^2)\Delta_+(x-x') = \delta(x-x'), \tag{4.40}$$

we see that ψ satisfies the inhomogeneous Dirac equation,

$$\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right)\psi = \eta. \tag{4.41}$$

Accordingly, the action is

$$W[\eta, \psi] = \int (dx) \left[\eta(x)\gamma^0\psi(x) + \mathcal{L}(x)\right], \tag{4.42}$$

where the Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\psi\gamma^0 \left[\gamma^\mu \frac{1}{i}\partial_\mu + m\right]\psi. \tag{4.43}$$

Which ψ the derivative acts on is immateria since

$$\psi\gamma^0\gamma^\mu\partial_\mu\psi = -\partial_\mu\psi\gamma^0\gamma^\mu\psi, \tag{4.44}$$

because $(\gamma^0\gamma^\mu)^T = \gamma^0\gamma^\mu$.

4.3 Feynman Rules

The route to spinor electrodynamics is through gauge transformations, as previously indicated. Implicit in such is the notion that the fermion has electric charge. That can be accomplished by using real fields with an extra two-fold multiplicity and an imaginary charge matrix (see homework), or, more conventionally, letting ψ be complex. Then the appropriate gauged action is

$$W[\eta, \overline{\eta}, J; \psi, A] = \int (dx) \left[\overline{\eta}\psi + \overline{\psi}\eta + J^\mu A_\mu + \mathcal{L}\right], \tag{4.45}$$

where

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \overline{\psi}\left(\gamma\frac{1}{i}D + m\right)\psi, \tag{4.46}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - ieA_\mu, \tag{4.47}$$

and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \overline{\eta} = \eta^\dagger\gamma^0, \quad \text{etc.} \tag{4.48}$$

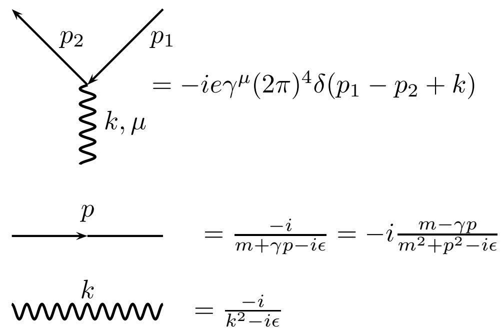

Figure 4.1: Feynman rules for QED

The startionary principle applied to field variations are the following field equations:

$$\delta\overline{\psi}: \quad \left(\gamma \frac{1}{i} D + m\right)\psi = \eta, \tag{4.49}$$

$$\delta\psi: \quad \overline{\psi}\left(-\gamma \frac{1}{i} \overleftarrow{D} + m\right) = \overline{\eta}, \tag{4.50}$$

$$\delta A: \quad \partial_\nu F^{\mu\nu} = j^\mu + J^\mu, \quad j^\mu = e\overline{\psi}\gamma^\mu\psi. \tag{4.51}$$

Note that the second of the these equations is simply the adjoint of the first, because the adjoint of γ^0 times (4.49) is

$$\psi^\dagger \gamma^0 \gamma^\mu \left(-\frac{1}{i}\partial_\mu - eA_\mu\right) = \eta^\dagger \gamma^0. \tag{4.52}$$

From this action, we can read off the Feynman rules for QED, as shown in Fig. 4.1. In addition, for external on-shell lines supply an appropriate wave-function: $e^\mu_{p\lambda}$, and for electron, either $u_{p\sigma}$ or $u^*_{p\sigma}\gamma^0$. Furthermore, a factor of -1 must be supplied for each closed Fermion loop, which is a consequence of

$$\begin{aligned} \eta^\dagger \gamma^0 G_+ \eta &= \eta^\dagger_\zeta (\gamma^0 G_+)_{\zeta\zeta'} \eta_{\zeta'} \\ &= -\eta_{\zeta'} \eta^\dagger_\zeta (\gamma^0 G_+)_{\zeta\zeta'} = -\mathrm{Tr}\, \eta\eta^\dagger \gamma^0 G_+. \end{aligned} \tag{4.53}$$

There are also minus signs coming from permutations of external lines. For example, the amputated graph shown in Fig. 4.2 has the value

$$-\int \frac{(dp)}{(2\pi)^4} \mathrm{Tr}\left(-ie\gamma^\mu \frac{-i}{m+\gamma p - i\epsilon}(-ie\gamma^\nu)\frac{-i}{m - \gamma(p+k) - i\epsilon}\right). \tag{4.54}$$

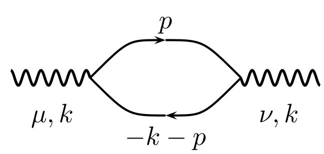

Figure 4.2: Lowest-order vacuum polarization graph