We must be careful here because

$$\partial_\mu \delta J^\mu = 0. \tag{4.15}$$

Thus we have the freedom to add to δW

$$0 = -\int (dx)\lambda(x)\partial_\mu \delta J^\mu(x), \tag{4.16}$$

where λ is an arbitrary function (Lagrange multiplier). Therefore, we conclude that

$$A_\mu(x) = \int (dx')D_+(x-x')J_\mu(x') + \partial_\mu \lambda(x). \tag{4.17}$$

λ represents the gauge freedom of the electromagnetic field. What equation does A_μ satisfy? Since $\partial_\mu A^\mu = \partial^2 \lambda$,

$$-\partial^2 A_\mu = J_\mu - \partial_\mu \partial_\nu A^\nu, \tag{4.18}$$

or

$$\partial^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\nu F_{\mu\nu} = J_\mu, \tag{4.19}$$

so we have recovered the Maxwell equations. Now we can follow the path we trod for a scalar field:

$$W[J] = \frac{1}{2}\int (dx)J^\mu A_\mu = \frac{1}{2}\int (dx)\partial_\nu F^{\mu\nu} A_\mu = \frac{1}{4}\int (dx) F^{\mu\nu}F_{\mu\nu}, \tag{4.20}$$

so we deduce the *action* form,

$$W[J,A] = \int (dx)\left[J^\mu A_\mu + \mathcal{L}\right], \tag{4.21}$$

where the Maxwell Lagrange density is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{4.22}$$

Because $\delta W = \int (dx)\delta J^\mu A_\mu$, we conclude that $W[J,A]$ is *stationary* with respect to field variations, from which follows the Maxwell equations.

4.2 Spin-1/2

Denote the spin-1/2 (Dirac) source by η, a four-component object. Is it correct to write simply ($\eta\gamma^0\psi$ is a scalar)

$$\langle 0_+|0_-\rangle_0^\eta = \exp\left[\frac{i}{2}\int (dx)(dx')\eta(x)\gamma^0\Delta_+(x-x')\eta(x')\right]? \tag{4.23}$$

No, because physically there are only two degrees of freedom for a spin-1/2 particle. Recalling that the rest-frame spinors are eigenvectors of γ^0 with eigenvalue +1, the restriction to the physical degrees of freedom is accomplished by inserting the projection operator

$$m - \gamma\frac{1}{i}\partial \to m - \gamma p \to m(1+\gamma^0), \tag{4.24}$$

where the first replacement occurs in momentum space, and the second in the rest frame. We take then as our vacuum persistence amplitude

$$\langle 0_+|0_-\rangle_0^\eta = \exp\left[\frac{i}{2}\int(dx)(dx')\eta(x)\gamma^0 G_+(x-x')\eta(x')\right], \tag{4.25}$$

where the spin-1/2 Green's function is

$$\begin{aligned} G_+(x-x') &= \left(m - \gamma^\mu\frac{1}{i}\partial_\mu\right)\Delta_+(x-x') \\ &= \int\frac{(dp)}{(2\pi)^4}\frac{m-\gamma p}{m^2+p^2-i\epsilon}, \end{aligned} \tag{4.26}$$

or in momentum space,

$$G_+(p) = \frac{1}{m+\gamma p - i\epsilon}. \tag{4.27}$$

But there is something rather strange about this construction. $\gamma^0 G_+(x-x')$ is totally antisymmetrical, under interchange of both space-time and Dirac coordinates:

$$\begin{aligned} \left[\gamma^0 G_+(x'-x)\right]^T &= \left[m\gamma^0 - \gamma^0\gamma^\mu\frac{1}{i}\partial'_\mu\right]^T\Delta_+(x'-x) \\ &= \left[-m\gamma^0 - \gamma^0\gamma^\mu\frac{1}{i}(-\partial_\mu)\right]\Delta_+(x-x') \\ &= -\gamma^0 G_+(x-x'). \end{aligned} \tag{4.28}$$

Correspondingly, the η_ζ (where $\zeta = 1,2,3,4$ is the Dirac index) must be *anti-commuting* numbers, that is elements of a Grassmann or exterior algebra:

$$\eta_\zeta(x)\eta_{\zeta'}(x') = -\eta_{\zeta'}(x')\eta_\zeta(x). \tag{4.29}$$

This makes the generating function nonzero:

$$\begin{aligned} W[\eta] &= \int(dx)(dx')\sum_{\zeta\zeta'}\eta_\zeta(x)\underbrace{\left[\gamma^0 G_+(x-x')\right]_{\zeta\zeta'}}_{-\left[\gamma^0 G_+(x'-x)\right]_{\zeta'\zeta}}\eta_{\zeta'}(x') \\ &= \int(dx)(dx')\sum_{\zeta\zeta'}\eta_{\zeta'}(x')\left[\gamma^0 G_+(x'-x)\right]_{\zeta'\zeta}\eta_\zeta(x). \end{aligned} \tag{4.30}$$

We now may easily verify that the probability requirement is satisfied:

$$\begin{aligned} 1 &\geq |\langle 0_+|0_-\rangle_0^\eta|^2 \\ &= \exp\left[-\mathrm{Re}\int (dx)(dx')\eta(x)\gamma^0\left(m-\gamma\frac{1}{i}\partial\right)\frac{1}{i}\Delta_+(x-x')\eta(x')\right], \end{aligned} \tag{4.31}$$

because $\eta_\zeta(x)\eta_{\zeta'}(x')$ is imaginary:

$$\begin{aligned} [\eta_\zeta(x)\eta_{\zeta'}(x')]^* &= \eta_{\zeta'}(x')^*\eta_\zeta(x)^* = \eta_{\zeta'}(x')\eta_\zeta(x) \\ &= -\eta_\zeta(x)\eta_{\zeta'}(x'). \end{aligned} \tag{4.32}$$

Further, γ^0 is imaginary, γ^μ/i is real, and

$$\mathrm{Re}\frac{1}{i}\Delta_+(x-x') = \mathrm{Re}\int d\tilde{p}\, e^{ip(x-x')}. \tag{4.33}$$

Acting on the latter is

$$m-\gamma\frac{1}{i}\partial \to m-\gamma p = 2m\sum_\lambda u_{p\lambda}u_{p\lambda}^*\gamma^0, \tag{4.34}$$

as we recall from (1.185a). Therefore the probability condition (unitarity) is satisfied

$$|\langle 0_+|0_-\rangle_0^\eta|^2 = \exp\left[-\sum_{p\lambda}\eta_{p\lambda}^*\eta_{p\lambda}\right] \leq 1, \tag{4.35}$$

where

$$\eta_{p\lambda} = \sqrt{2m\,d\tilde{p}}\,u_{p\lambda}^*\gamma^0\eta(p). \tag{4.36}$$

4.2.1 Fields

As before, the field ψ is defined by

$$\delta W[\eta] = \int (dx)\delta\eta(x)\gamma^0\psi(x) = \int (dx)\psi(x)\gamma^0\delta\eta(x), \tag{4.37}$$

where the two forms are equal since the antisymmetry of γ^0 compensates for the anticommutativity of $\delta\eta$ and ψ. For noninteracting fermions,

$$W[\eta] = \frac{1}{2}\int (dx)(dx')\eta(x)\gamma^0 G_+(x-x')\eta(x'), \tag{4.38}$$

so

$$\psi(x) = \int (dx')G_+(x-x')\eta(x'), \tag{4.39a}$$

$$\psi(x)\gamma^0 = \int (dx')\eta(x')\gamma^0 G_+(x'-x), \tag{4.39b}$$

which follow from the alternative definitions in (4.37) and are directly related by the total antisymmetry of $\gamma^0 G_+(x-x')$. Because the Dirac Green's function satisfies

$$\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right) G_+(x-x') = (m^2 - \partial^2)\Delta_+(x-x') = \delta(x-x'), \tag{4.40}$$

we see that ψ satisfies the inhomogeneous Dirac equation,

$$\left(\gamma^\mu \frac{1}{i}\partial_\mu + m\right) \psi = \eta. \tag{4.41}$$