Chapter 4

Quantum Electrodynamics

4.1 Spin One and the Photon

Let us now turn to spin 1. We expect that the source for a spin-1 particle is a vector J_μ. However, were we to adopt, as the vacuum persistence amplitude,

$$\langle 0_+|0_-\rangle_0^J = \exp\left[\frac{i}{2}\int (dx)(dx')J^\mu(x)\Delta_+(x-x')J_\mu(x')\right], \tag{4.1}$$

we would have a conflict with unitarity:

$$\begin{aligned}|\langle 0_+|0_-\rangle_0^J|^2 &= \exp\left[-\int (dx)(dx')J^\mu(x)\mathrm{Re}\,\frac{1}{i}\Delta_+(x-x')J_\mu(x)\right] \\ &= \exp\left[-\int d\tilde{p} J^\mu(-p)J_\mu(p)\right] \le 1,\end{aligned} \tag{4.2}$$

which will be true only if

$$\int d\tilde{p}\left[|\mathbf{J}(p)|^2 - |J^0(p)|^2\right] \ge 0. \tag{4.3}$$

We must therefore suppress the time component of J_μ in a relativistically invariant manner. We can do this by use of the timelike vector p^μ. That is, in $J^{\mu *}(p)g_{\mu\nu}J^\nu(p)$ we replace $g_{\mu\nu} \to g_{\mu\nu} + p_\mu p_\nu/m^2$. This does the trick, for in the rest frame of p^μ,

$$p^\mu = (m, \mathbf{0}): \quad g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} = \begin{cases} \delta_{kl}, & \mu = k, \nu = l, \\ 0, & \text{otherwise.}\end{cases} \tag{4.4}$$

So

$$J^\mu(p)^*\left(g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)J^\nu(p) = |\mathbf{J}(p)|^2 \ge 0. \tag{4.5}$$

The correct vacuum persistence amplitude is then

$$\langle 0_+|0_-\rangle_0^J = \exp\left[\frac{i}{2}\int (dx)(dx')J^\mu(x)\Delta_{+\mu\nu}(x-x')J^\nu(x')\right], \tag{4.6}$$

where

$$\Delta_{+\mu\nu}(x-x') = \left(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_+(x-x'). \tag{4.7}$$

The projection operator restricts the source to the three physical polarization states, which we make explicit by writing

$$g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} = \sum_{\lambda=1}^{3} e_{\mu,p\lambda}e^*_{\nu,p\lambda}, \tag{4.8}$$

where the polarization vectors satisfy

$$p^\mu e_{\mu,p\lambda} = 0, \quad e^\mu_{p\lambda}e_{\mu,p\lambda'} = \delta_{\lambda\lambda'}. \tag{4.9}$$

The photon, however, is a massless particle. We can obtain the appropriate vacuum amplitude expression by a suitable careful limit. Let $\partial_\mu J^\mu(x) = m^2 K(x)$. Then

$$\begin{aligned}\langle 0_+|0_-\rangle &= \exp\left[\frac{i}{2}\int(dx)(dx')\big(J^\mu(x)\Delta_+(x-x')J_\mu(x')\right.\\ &\qquad\left.+K(x)\Delta_+(x-x')K(x')\big)\right],\end{aligned} \tag{4.10}$$

amd now take the massless limit, $m\to 0$:

$$\lim_{m\to 0}\Delta_+(x-x') = D_+(x-x') = \int\frac{(dk)}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2-i\epsilon}, \tag{4.11}$$

and $\partial_\mu J^\mu = 0$. The three degrees of freedom of the massive vector become the two degrees of freedom of the massless vector, and the one degree of freedom of an independent, uncoupled, massless scalar. We may disregard the latter. So the vacuum persistence amplitude for a photon is

$$\langle 0_+|0_-\rangle_0^{J,m=0} = \exp\left[\frac{i}{2}\int(dx)(dx')J^\mu(x)D_+(x-x')J_\mu(x')\right], \tag{4.12}$$

where $\partial_\mu J^\mu = 0$. This guarantees $|\langle 0_+|0_-\rangle^J|^2 \le 0$, as you will show in the homework. We see here the requirement of current conservation – a photon must couple to a conserved charge.

If we call for the photon

$$W[J] = \frac{1}{2}\int(dx)(dx')J^\mu(x)D_+(x-x')J_\mu(x'), \tag{4.13}$$

we define the (vector potential) field by

$$\delta W[J] = \int(dx)\delta J^\mu(x)A_\mu(x). \tag{4.14}$$

We must be careful here because

$$\partial_\mu \delta J^\mu = 0. \tag{4.15}$$

Thus we have the freedom to add to δW

$$0 = -\int (dx)\lambda(x)\partial_\mu \delta J^\mu(x), \tag{4.16}$$

where λ is an arbitrary function (Lagrange multiplier). Therefore, we conclude that

$$A_\mu(x) = \int (dx')D_+(x-x')J_\mu(x') + \partial_\mu \lambda(x). \tag{4.17}$$

λ represents the gauge freedom of the electromagnetic field. What equation does A_μ satisfy? Since $\partial_\mu A^\mu = \partial^2\lambda$,

$$-\partial^2 A_\mu = J_\mu - \partial_\mu \partial_\nu A^\nu, \tag{4.18}$$

or

$$\partial^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\nu F_{\mu\nu} = J_\mu, \tag{4.19}$$

so we have recovered the Maxwell equations. Now we can follow the path we trod for a scalar field:

$$W[J] = \frac{1}{2}\int (dx)J^\mu A_\mu = \frac{1}{2}\int (dx)\partial_\nu F^{\mu\nu} A_\mu = \frac{1}{4}\int (dx)F^{\mu\nu}F_{\mu\nu}, \tag{4.20}$$

so we deduce the *action* form,

$$W[J,A] = \int (dx)\left[J^\mu A_\mu + \mathcal{L}\right], \tag{4.21}$$

where the Maxwell Lagrange density is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{4.22}$$

Because $\delta W = \int (dx)\delta J^\mu A_\mu$, we conclude that $W[J,A]$ is *stationary* with respect to field variations, from which follows the Maxwell equations.