3.4 Example: Two-Point Function

The lowest-order graphs contributing to the two-point function are shown in Fig. 3.3. These graphs are immediately translated into the following expressions:

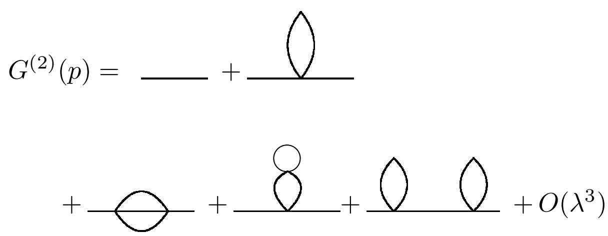


Figure 3.3: Graphs contributing to the two-point Green's function through $O(\lambda^2)$.

$$\begin{aligned}
G^{(2)}(p) \;=\; & \frac{-i}{p^2+m^2-i\epsilon} - i\frac{\lambda}{2}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
& + \frac{(-i\lambda)^2}{3!}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{(dl')}{(2\pi)^4}\frac{(dl'')}{(2\pi)^4} \\
& \times\frac{(2\pi)^4\delta(l+l'+l''-p)(-i)^3}{(l^2+m^2-i\epsilon)(l'^2+m^2-i\epsilon)(l''^2+m^2-i\epsilon)}\frac{-i}{p^2+m^2-i\epsilon} \\
& + \frac{(-i\lambda)^2}{4}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{(-i)^2}{(l^2+m^2-i\epsilon)^2} \\
& \times\int\frac{(dl')}{(2\pi)^4}\frac{-i}{l'^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
& + \frac{(-i\lambda)^2}{4}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
& \times\int\frac{(dl')}{(2\pi)^4}\frac{-i}{l'^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon}.
\end{aligned} \tag{3.38}$$

Note that we can sum the subset of graphs that consist of iterations of a single bubble, as shown in Fig. 3.4. If we define the "mass operator" or "self-energy part" as the one-loop graph with the external propagators removed, or "amputated,"

$$i\Sigma = -i\frac{\lambda}{2}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}, \tag{3.39}$$

the sum of the graphs indicated in Fig. 3.4 is

$$\begin{aligned}
G'(p) \;=\; & \frac{-i}{p^2+m^2-i\epsilon} + \frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon} \\
& + \frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon} + \ldots
\end{aligned}$$

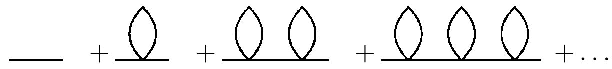
$G'(p) =$ 

Figure 3.4: Iterated one-loop contribution to the "vacuum polarization," leading to the one-loop "self energy."

$$= \frac{-i}{p^2+m^2-\Sigma-i\epsilon}, \tag{3.40}$$

due to the geometric series,

$$\frac{1}{x-y} = \frac{1}{x(1-y/x)} = \frac{1}{x}\sum_{n=0}^{\infty}\left(\frac{y}{x}\right)^n, \tag{3.41}$$

valid if $y/x < 1$. (Here this is formally satisfied, since we are assuming that λ is small.) Here we see that because in this case Σ is constant, it indeed amounts to a shift in the mass, hence the name.

However, we notice that Σ is divergent: If we pass to Euclidean space, as in Problem 4.3, and introduce polar coordinates, $(dl)_E = d\Omega\, l^3 dl$, where Ω is the volume of a unit 4-sphere, the Euclidean form of Σ is

$$\int (dl)_E \frac{1}{l_E^2+m^2} = \frac{\Omega}{2}\int_0^{\Lambda^2}\frac{dl^2\, l^2}{l^2+m^2} = \frac{\Omega}{2}\left(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right). \tag{3.42}$$

Here, we have inserted an ultraviolet momentum cutoff Λ. As $\Lambda\to\infty$, the integral is quadratically divergent.

If we include the third graph in Fig. 3.3, the so-called "sunset" graph, the self-energy part acquires momentum dependence:

$$\Sigma(p) = A + Bp^2, \tag{3.43}$$

where A is quadratically divergent, $A\sim\Lambda^2$, and B is logarithmically divergent, $B\sim\ln\Lambda^2/m^2$. (There remains finite p^2 dependence.) Such a contribution not only changes the location of the pole (by an infinite amount), but changes its residue as well:

$$\frac{1}{p^2+m^2-\Sigma(p)} = \frac{1}{p^2+m^2-A-Bp^2} = \frac{1}{1-B}\frac{1}{p^2+\frac{m^2-A}{1-B}}. \tag{3.44}$$

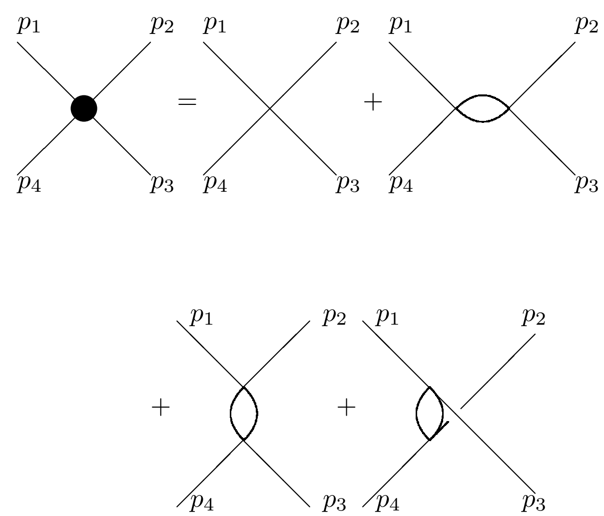


Figure 3.5: Graphs through order λ^2 contributing to the four-point function. In addition, there are propagator corrections to the external lines.

3.5 Renormalization Theory

The four-point function also contains divergences. The graphs, through second order, are shown in Fig. 3.5. Here, for example, the second graph is, if we amputate the external legs,

$$\frac{(-i\lambda)^2}{2}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}\frac{-i}{(l-p_1-p_4)^2+m^2-i\epsilon}. \tag{3.45}$$

By power counting, it is clear that this is logarithmically divergent:

$$\sim\int^\Lambda\frac{l^3\,dl}{l^4}\sim\ln\frac{\Lambda}{m}. \tag{3.46}$$

Comparing with the lowest order graph, $-i\lambda$, we see that this results in an infinite shift in the coupling constant. These shifts in the parameters of the theory we call *renormalization.* What makes this sensible is that there are only three infinite constants in the theory, which may be taken to be

- m_0, the bare mass,
- λ_0, the bare coupling constant,
- Z, the wavefunction renormalization constant.

The Lagrangian, then, may be expressed either in terms of bare quantities, including the bare field ϕ_0,

$$\mathcal{L}=-\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0-\frac{1}{2}m_0^2\phi_0^2-\frac{\lambda_0}{4!}\phi_0^4, \tag{3.47}$$

or in terms of renormalizable ones

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \mathcal{L}_{\rm ct}, \tag{3.48}$$

where m and λ are the renormalized ("physical") mass and coupling constants, respectively, and the "counterterm" Lagrangian is

$$\mathcal{L}_{\rm ct} = -\frac{1}{2}C\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{\delta\lambda}{4!}\phi^4, \tag{3.49}$$

where C, δm, and $\delta\lambda$ may be expressed as power series in λ, with divergent coefficients. *Note that only structures present in the original Lagrangian appear in $\mathcal{L}_{\rm ct}$.* This is what we mean by a *renormalizable theory*;—A nonrenormalizable theory would require the introduction of an infinite number of counterterms, with structures that do not appear in $\mathcal{L}_0$. By comparing the two forms of $\mathcal{L}$ above we see

$$\begin{aligned} \phi_0 &= (1+C)^{1/2}\phi \equiv Z^{1/2}\phi, && (3.50a)\\ m_0^2 Z &= m^2 + \delta m^2, && (3.50b)\\ \lambda_0 Z^2 &= \lambda + \delta\lambda, && (3.50c) \end{aligned}$$

or

$$\begin{aligned} Z &= 1 + C, && (3.51a)\\ m_0^2 &= (m^2 + \delta m^2)Z^{-1}, && (3.51b)\\ \lambda_0 &= (\lambda + \delta\lambda)Z^{-2}. && (3.51c) \end{aligned}$$

As we've seen above,

$$\begin{aligned} \lambda_0 &= \lambda + O(\lambda^2), && (3.52a)\\ \frac{m_0}{m} &= 1 + O(\lambda), && (3.52b)\\ Z &= 1 + O(\lambda^2), && (3.52c) \end{aligned}$$

where the last comes from the sunset graph in Fig. 3.3.

All infinities in the theory can be absorbed in λ_0, m_0, and Z; and finite, renormalized Green's functions are given by

$$G^{(n)}(p_1,\dots,p_n;\lambda,m) = Z^{-n/2}G_0^{(n)}(p_1,\dots,p_n;\lambda_0,m_0). \tag{3.53}$$

Let us see how this works through second order. By making the coupling constant explicit, we have for the bare two-point function,

$$G_0^{(2)}(p) = \frac{1}{1-\lambda_0^2 B}\frac{-i}{p^2 + \frac{m_0^2 - \lambda_0 A_1 - \lambda_0^2 A_2}{1-\lambda_0^2 B}}, \tag{3.54}$$

up to finite corrections, while the renormalized Green's function (to this order) is simply

$$G^{(2)}(p) = Z^{-1}G_0^{(2)}(p) = \frac{-i}{p^2+m^2} - i\int_{(3m)^2}^{\infty} dM^2\,\frac{a(M^2)}{p^2+M^2}, \tag{3.55}$$

the latter term, in terms of a spectral density $a(M^2)$, expressing a branch line starting at the three-particle threshold, we conclude that

$$Z = 1 + \lambda^2 B + O(\lambda^3), \tag{3.56}$$

using (3.52a). Then from

$$m^2 = \frac{m_0^2 - \lambda_0 A_1 - \lambda_0^2 A_2}{1-\lambda_0^2 B} = Z(m_0^2 - \lambda_0 A_1 - \lambda_0^2 A_2), \tag{3.57}$$

or

$$Zm_0^2 = m^2 + \delta m^2, \quad \delta m^2 = Z(\lambda_0 A_1 + \lambda_0^2 A_2) \approx \lambda_0 A_1 + \lambda_0^2 A_2. \tag{3.58}$$

Now the bare four-point function is

$$G_0^{(4)} = \prod_{i=1}^{4} \frac{-i}{p_i^2 + m_0^2 - \Sigma}(-i\lambda_0)\left[1 + \lambda_0\left(D + E(s,t,u)\right)\right], \tag{3.59}$$

so the renormalized four-point function is

$$\begin{aligned} G^{(4)} = Z^{-2}G_0^{(4)} &= \prod_{i=1}^{4} \frac{-i}{p^2+m^2} Z^2(-i\lambda_0)\left[1+\lambda_0\left(D+E(s,t,u)\right)\right] \\ &= \prod_{i=1}^{4} \frac{-i}{p^2+m^2}(-i\lambda)\left[1+\lambda E(s,t,u)\right]. \end{aligned} \tag{3.60}$$

Here we have introduced the so-called Mandelstam variables,

$$s = (p_1+p_2)^2, \quad t = (p_1+p_3)^2, \quad u = (p_1+p_4)^2. \tag{3.61}$$

Therefore,

$$Z^2\lambda_0(1+\lambda D) = \lambda, \tag{3.62}$$

or

$$Z^2\lambda_0 = \lambda(1-\lambda D) = \lambda + \delta\lambda. \tag{3.63}$$

Thus we have expressed the counterterms in terms of power series in the renormalized coupling constant, with infinite coefficients:

$$Z = 1 + \lambda^2 B, \tag{3.64a}$$
$$\delta\lambda = -\lambda^2 D, \tag{3.64b}$$
$$\delta m^2 = \lambda A_1 + \lambda^2(A_2 - DA_1). \tag{3.64c}$$