3.3 Perturbation Theory

How do we introduce interactions? We start things off by inserting a (primitive) interaction term in W:

$$W = \int (dx) \left[K\phi - \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi + m^2\phi^2) - \frac{\lambda}{4!}\phi^4 \right], \tag{3.28}$$

where λ is a dimensionless coupling constant, because the dimension of ϕ is

$$[\phi] = M = L^{-1}. \tag{3.29}$$

This will induce further terms in W, which we will finally write as

$$iW[K] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int (dx_1)(dx_2)\dots(dx_n) K(x_1)\dots K(x_n) G_c^{(n)}(x_1,\dots,x_n), \tag{3.30}$$

where the $G_c^{(n)}$ are the "connected Green's functions" of the theory. For a free theory ($\lambda = 0$) the only nonvanishing G_c is

$$G_{c,0}^{(2)}(x,y) = -i\Delta_+(x-y). \tag{3.31}$$

Of course, we cannot compute $W[K]$ exactly. We will describe an approximation scheme, based on the smallness of λ called (weak-coupling) perturbation theory. Once $W[K]$ is computed (to some accuracy) we can obtain the Green's functions of the theory by functional differentiation:

$$G_c^{(n)}(x_1,\dots,x_n) = i^{1-n} \frac{\delta^n}{\delta K(x_1)\dots\delta K(x_n)} W[K]\bigg|_{K=0}. \tag{3.32}$$

The approximation method we will describe—perturbation theory—is based on the assumption that $\lambda \ll 1$, and thereby we may develop a power series expansion for the Green's functions in terms of λ. An easy way to see how to do this is to write down the equation of motion which follows from W in the form (3.28):

$$(-\partial^2 + m^2)\phi + \frac{\lambda}{3!}\phi^3 = K, \tag{3.33}$$

Thus, the interaction term acts like an effective source:

$$K\bigg|_{\text{eff}} = -\frac{\lambda}{3!}\phi^3. \tag{3.34}$$

Now in momentum space

$$\begin{aligned}\int (dx) K(x)\phi(x) &= \int (dx) \int \frac{(dP)}{(2\pi)^4} e^{iPx} K(P) \int \frac{(dQ)}{(2\pi)^4} e^{iQx} \phi(Q) \\ &= \int \frac{(dP)}{(2\pi)^4} K(P)\phi(-P), \end{aligned} \tag{3.35}$$

so if we insert the effective source (3.34), we get

$$\begin{aligned} K(P)\phi(-P)\Big|_{\text{eff}} &= -\frac{\lambda}{3!}\int (dx)e^{-iPx}\int \frac{(dp_1)}{(2\pi)^4}e^{ip_1x}\phi(p_1) \\ &\quad\times \int \frac{(dp_2)}{(2\pi)^4}e^{ip_1x}\phi(p_2)\int \frac{(dp_3)}{(2\pi)^4}e^{ip_1x}\phi(p_3)\phi(-P) \\ &= -\lambda\int \frac{(dp_1)(dp_2)(dp_3)}{3!(2\pi)^{12}}\phi(p_1)\phi(p_2)\phi(p_3)\phi(-P) \\ &\quad\times(2\pi)^4\delta(P-p_1-p_2-p_3), \end{aligned} \tag{3.36}$$

where the 3! is the number of ways of permuting the three momentum labels p_1, p_2, p_3.

From this we can infer the momentum-space Feynman rules. From the term in iW representing the term corresponding to the exchange of a particle between effective sources,

$$i\int (dx)(dy)K_1\Big|_{\text{eff}}(x)\Delta_+(x-y)K_2\Big|_{\text{eff}}(y), \tag{3.37}$$

each propagator is represented by a line as shown in Fig. 3.1. From (3.36), each

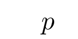
$\frac{-i}{p^2+m^2-i\epsilon} =$ 

Figure 3.1: Propagator line. The momentum carried by the line is p.

vertex is represented by the graph shown in Fig. 3.2. To obtain $G_c^{(n)}$ to order λ^k,

- Draw all connected graphs with n external lines having up to k vertices, which are topologically distinct.
- Conserve momentum at each vertex by supplying the factor $(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4)$, all momentum flowing into the vertex.
- Integrate over loop momenta, $\int(dl)/(2\pi)^4$.
- Divide by the symmetry number.

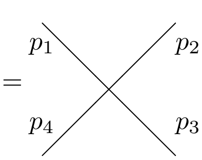
$-i\lambda =$ 

Figure 3.2: Vertex graph. Each momentum p_i is regarded as incoming.

3.4 Example: Two-Point Function

The lowest-order graphs contributing to the two-point function are shown in Fig. 3.3. These graphs are immediately translated into the following expressions:

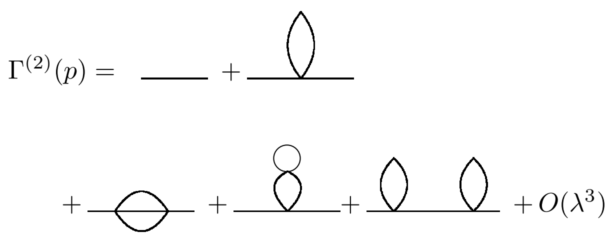


Figure 3.3: Graphs contributing to the two-point Green's function through $O(\lambda^2)$.

$$
\begin{aligned}
G^{(2)}(p) \;=\; & \frac{-i}{p^2+m^2-i\epsilon} - i\frac{\lambda}{2}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
&+\frac{(-i\lambda)^2}{3!}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{(dl')}{(2\pi)^4}\frac{(dl'')}{(2\pi)^4} \\
&\times\frac{(2\pi)^4\delta(l+l'+l''-p)(-i)^3}{(l^2+m^2-i\epsilon)(l'^2+m^2-i\epsilon)(l''^2+m^2-i\epsilon)}\frac{-i}{p^2+m^2-i\epsilon} \\
&+\frac{(-i\lambda)^2}{4}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{(-i)^2}{(l^2+m^2-i\epsilon)^2} \\
&\times\int\frac{(dl')}{(2\pi)^4}\frac{-i}{l'^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
&+\frac{(-i\lambda)^2}{4}\frac{-i}{p^2+m^2-i\epsilon}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{l^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon} \\
&\times\int\frac{(dl')}{(2\pi)^4}\frac{-i}{l'^2+m^2-i\epsilon}\frac{-i}{p^2+m^2-i\epsilon}. \qquad (3.38)
\end{aligned}
$$

Note that we can sum the subset of graphs that consist of iterations of a single bubble, as shown in Fig. 3.4. If we define the "mass operator" or "self-energy part" as the one-loop graph with the external propagators removed, or "amputated,"

$$
i\Sigma = -i\frac{\lambda}{2}\int\frac{(dl)}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}, \qquad (3.39)
$$

the sum of the graphs indicated in Fig. 3.4 is

$$
\begin{aligned}
G'(p) \;=\; & \frac{-i}{p^2+m^2-i\epsilon}+\frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon} \\
&+\frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon}i\Sigma\frac{-i}{p^2+m^2-i\epsilon}+\dots
\end{aligned}
$$

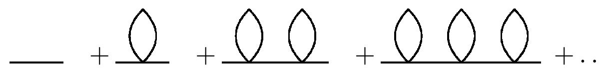


Figure 3.4: Iterated one-loop contribution to the “vacuum polarization,” leading to the one-loop “self energy.”

$$= \frac{-i}{p^2 + m^2 - \Sigma - i\epsilon}, \tag{3.40}$$

due to the geometric series,

$$\frac{1}{x-y} = \frac{1}{x(1-y/x)} = \frac{1}{x}\sum_{n=0}^{\infty}\left(\frac{y}{x}\right)^n, \tag{3.41}$$

valid if $y/x < 1$. (Here this is formally satisfied, since we are assuming that λ is small.) Here we see that because in this case Σ is constant, it indeed amounts to a shift in the mass, hence the name.

However, we notice that Σ is divergent: If we pass to Euclidean space, as in Problem 4.3, and introduce polar coordinates, $(dl)_E = d\Omega\, l^3 dl$, where Ω is the volume of a unit 4-sphere, the Euclidean form of Σ is

$$\int (dl)_E \frac{1}{l_E^2 + m^2} = \frac{\Omega}{2}\int_0^{\Lambda^2} \frac{dl^2\, l^2}{l^2 + m^2} = \frac{\Omega}{2}\left(\Lambda^2 - m^2 \ln\frac{\Lambda^2}{m^2}\right). \tag{3.42}$$

Here, we have inserted an ultraviolet momentum cutoff Λ. As $\Lambda \to \infty$, the integral is quadratically divergent.

If we include the third graph in Fig. 3.3, the so-called “sunset” graph, the self-energy part acquires momentum dependence:

$$\Sigma(p) = A + Bp^2, \tag{3.43}$$

where A is quadratically divergent, $A \sim \Lambda^2$, and B is logarithmically divergent, $B \sim \ln \Lambda^2/m^2$. Such a contribution not only changes the location of the pole (by an infinite amount), but changes its residue as well:

$$\frac{1}{p^2 + m^2 - \Sigma(p)} = \frac{1}{p^2 + m^2 - A - Bp^2} = \frac{1}{1-B}\frac{1}{p^2 + \frac{m^2 - A}{1-B}}. \tag{3.44}$$