Chapter 3

Perturbation Theory—Feynman Rules

3.1 Feynman Propagator

For simplicity, let us consider spin-0 particles first. Consider a free particle coupled to an external source $K(x)$, described by an inhomogeneous Klein-Gordon equation

$$(-\partial^2 + m^2)\phi(x) = K(x). \tag{3.1}$$

The corresponding action is

$$W = -\int (dx) \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + m^2\phi^2) - K\phi\right], \tag{3.2}$$

for $\delta W/\delta\phi$ yields the Klein-Gordon equation (3.1). Let us introduce a Green's function (or propagation function or propagator), given by $\Delta_+(x - x')$, which satisfies the differential equation

$$(-\partial^2 + m^2)\Delta_+(x - x') = \delta(x - x'), \tag{3.3}$$

subject to boundary conditions (initial conditions) yet to be specified. In momentum space, the solution to this Green's function equation can be written as

$$\Delta_+(x - x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2 + m^2}, \tag{3.4}$$

where

$$p(x - x') = p \cdot (x - x') = p_\mu (x - x')^\mu, \quad (dp) = dp_0\,(d\mathbf{p}), \tag{3.5}$$

where the appearance of dp_0 rather than dp^0 is suggested by

$$\int_{-\infty}^{\infty} dp_0\, e^{ip_0(x-x')^0} = 2\pi\delta[(x - x')^0]. \tag{3.6}$$

But the above integral (3.4) does not exist, because of the singularity at $p^2 + m^2 = 0$, that is when the relation between the particle's energy and momentum is satisfied ("on the mass shell"). The Green's function we want is actually given by

$$\Delta_+(x-x') = \int \frac{(dp)}{(2\pi)^4} \frac{e^{ip(x-x')}}{p^2+m^2-i\epsilon}, \tag{3.7}$$

as we shall see by carrying out the p_0 integration:

$$I = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{ip_0(x-x')^0}}{-(p_0-\omega_p+i\epsilon)(p_0+\omega_p-i\epsilon)}, \tag{3.8}$$

where $\omega_p = +\sqrt{\mathbf{p}^2+m^2}$. Now by Jordan's lemma, if $(x-x')^0 > 0$ we must close the contour in the upper half plane, while if $(x-x')^0 < 0$ we must close the contour in the lower half plane. This means that in the former case the pole at $-\omega_p + i\epsilon$ is enclosed, while in the latter case the pole at $\omega_p - i\epsilon$ is enclosed, so

$$\begin{aligned} I &= \begin{cases} \frac{1}{2\pi} 2\pi i \frac{1}{2\omega_p} e^{-i\omega_p(x-x')^0}, & (x-x')^0 > 0, \\ \frac{1}{2\pi} 2\pi i \frac{1}{2\omega_p} e^{+i\omega_p(x-x')^0}, & (x-x')^0 < 0, \end{cases} \\ &= \frac{i}{2\omega_p} e^{-i\omega_p|x^0-x'^0|} \end{aligned} \tag{3.9}$$

so

$$\Delta_+(x-x') = \begin{cases} i\int d\tilde{p}\, e^{ip(x-x')}, & (x-x')^0 > 0, \\ i\int d\tilde{p}\, e^{-ip(x-x')}, & (x-x')^0 < 0, \end{cases} \tag{3.10}$$

where $p^0 = -p_0 = \omega_p = +\sqrt{\mathbf{p}^2+m^2}$, and

$$d\tilde{p} = \frac{(d\mathbf{p})}{(2\pi)^3 2\omega_p} = \int \frac{(dp)}{(2\pi)^4}(2\pi)\delta(p^2+m^2)\eta(p^0), \tag{3.11}$$

where the Heaviside step function (note that capital η is H)

$$\eta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} . \tag{3.12}$$

The second equality in (3.11) follows from

$$\int_{-\infty}^{\infty} dp_0\, \delta\left((p^0)^2 - \omega_p^2\right)\eta(p^0) = \frac{1}{2\omega_p}. \tag{3.13}$$

The integration element in (3.11) is called the relativistically invariant three-momentum measure, or phase-space measure. Thus we have the following form for Δ_+:

$$\Delta_+(x-x') = \Delta_+(x'-x) = i\int d\tilde{p}\, e^{ip(x-x')}, \quad (x-x')^0 > 0. \tag{3.14}$$

That is, we seek the solution to the Green's function equation (3.3) which is symmetric and which corresponds to *positive* energy excitations moving forward in time (hence the + subscript). For further details, and connection to Euclidean space, see homework.

3.2 Vacuum Persistence Amplitude

Now we are ready to make the connection with quantum mechanics. We will take as an ansatz for noninteracting scalar particles the following expression for the *vacuum persistence amplitude*:

$$\langle 0,+\infty|0,-\infty\rangle_0^K = Z_0[K] = e^{iW[K]/\hbar}, \tag{3.15}$$

where

$$W[K] = \frac{1}{2}\int (dx)(dx')K(x)\Delta_+(x-x')K(x'). \tag{3.16}$$

Here the vacuum persistence amplitude $\langle 0,+\infty|0,-\infty\rangle_0^K$ is the probability amplitude that a no particle state before the sources are turned on remains a no particle state after the sources are turned off. This may be readily proved from other formulations of quantum field theory, but it is convenient here to simply adopt it as a starting point. We will content ourselves first by establishing a slight check of consistency:

$$|\langle 0,+\infty|0,-\infty\rangle_0^K|^2 = e^{-2\,\mathrm{Im}\,W}, \tag{3.17}$$

where

$$\begin{aligned}\mathrm{Im}\,W &= \frac{1}{2}\mathrm{Re}\int (dx)(dx')K(x)\int d\tilde{p}\,e^{ip(x-x')}K(x') \\ &= \frac{1}{2}\mathrm{Re}\int d\tilde{p}\,K(-p)K(p) = \frac{1}{2}\int d\tilde{p}\,|K(p)|^2 > 0,\end{aligned} \tag{3.18}$$

where the Fourier transform of the source function is

$$K(p) = \int (dx)e^{-ipx}K(x), \quad K(p)^* = K(-p). \tag{3.19}$$

Here we have taken $K(x)$ as real—a complex $K(x)$ would encode some additional property, such as charge. Equation (3.18) shows that probability is conserved, in that the probability of remaining in the vacuum state is less than unity.

Note that W here is actually equal to the action previously introduced, by virtue of the equations of motion:

$$\begin{aligned}W &= \frac{1}{2}\int (dx)(dx')K(x)\Delta_+(x-x')K(x') \\ &= \frac{1}{2}\int (dx)K(x)\phi(x) = \frac{1}{2}\int (dx)\left[(-\partial^2+m^2)\phi(x)\right]\phi(x) \\ &= \frac{1}{2}\int (dx)(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) \\ &= \int (dx)\left[K\phi - \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi+m^2\phi^2)\right],\end{aligned} \tag{3.20}$$

where in the last expression we simply took twice the second form minus the fourth form. Because

$$\phi(x) = \int (dx')\Delta_+(x-x')K(x') = \frac{\delta W[K]}{\delta K(x)}, \tag{3.21}$$

only in the last form of (3.20) do we have the action principle:

$$\frac{\delta W}{\delta\phi} = 0 \Rightarrow (-\partial^2 + m^2)\phi = K. \tag{3.22}$$

As a further indication of the validity of our ansatz, let us consider the interaction between two *static* sources, $K_1(x)$ and $K_2(x)$, described by

$$\exp\left[i\int (dx)(dx')K_1(x)\Delta_+(x-x')K_2(x)\right]. \tag{3.23}$$

Because the sources are static, $K_{1,2}(x) = K_{1,2}(\mathbf{x})$, and the exponent here becomes

$$i\int (d\mathbf{x})(d\mathbf{x}')K_1(\mathbf{x})\int dt\,dt'\,\Delta_+(x-x')K_2(\mathbf{x}), \tag{3.24}$$

where the time integral over the propagation function is

$$\begin{aligned}\int dt\,dt'\,\Delta_+(x-x') &= T\int d(t-t')\Delta_+(x-x')\\
&= T\int d(t-t')\int\frac{(dp)}{(2\pi)^4}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')+ip_0(t-t')}}{p^2+m^2-i\epsilon}\\
&= T\int\frac{(d\mathbf{p})}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{p}^2+m^2}\\
&= T\frac{1}{(2\pi)^2}\int_0^\infty p^2\,dp\frac{1}{p^2+m^2}\int_{-1}^1 d(\cos\theta)e^{ipR\cos\theta}\\
&= T\frac{1}{(2\pi)^2}\frac{1}{2}\int_{-\infty}^\infty\frac{p\,dp}{iR}\frac{1}{p^2+m^2}\left(e^{ipR}-e^{-ipR}\right)\\
&= \frac{T}{(2\pi)^2}\frac{1}{2}\frac{2\pi}{R}\left[\frac{1}{2im}ime^{-mR}+\frac{1}{-2im}(-im)e^{-mR}\right]\\
&= \frac{T}{4\pi}\frac{1}{R}e^{-mR},\end{aligned} \tag{3.25}$$

where T represents the long time that the sources are active, and $R = |\mathbf{x}-\mathbf{x}'|$. Thus, we can write the vacuum persistence amplitude for static sources as

$$\langle 0,+\infty|0,-\infty\rangle = e^{-iE_{\text{int}}T}, \tag{3.26}$$

where the energy of interaction between the two sources is

$$E_{\text{int}} = -\int (d\mathbf{x})(d\mathbf{x}')K_1(\mathbf{x})\frac{1}{4\pi}\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}K_2(\mathbf{x}). \tag{3.27}$$

This is the Yukawa interaction energy between two static source distributions. The minus sign indicates the attractive nature of the force. (As we shall see, for vector interactions the force is repulsive.)