just as advertized.

This "gauging" procedure leads to the *minimal* interaction between electron and photon, but there can be further interactions, as long as they are gauge invariant. The *anomalous magnetic moment* term

$$W' = -\frac{\kappa}{2}\int (dx)\overline{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi \tag{2.44}$$

is gauge invariant by itself, because F appears and no derivatives act on ψ. This adds to the Dirac equation the term

$$\frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}\psi = \kappa\mathbf{\Sigma}\cdot\mathbf{H}\psi \tag{2.45}$$

for a pure magnetic field. If such an extra term were present, the net magnetic moment would be [see (1.39)]

$$\boldsymbol{\mu} = \left(\frac{e}{2m} + \kappa\right)\mathbf{\Sigma} = \frac{e}{2m}\frac{g}{2}\mathbf{\Sigma}, \tag{2.46}$$

which is to say

$$\kappa = \frac{e}{2m}\left(\frac{g-2}{2}\right). \tag{2.47}$$

However, experimentally, for the electron, g is very nearly 2:

$$\frac{g_e}{2} = 1.001159652187(4), \tag{2.48}$$

and similarly for the muon:

$$\frac{g_\mu}{2} = 1.0011659160(6). \tag{2.49}$$

Does this mean that a small κ value should be inserted ad hoc? No! We will calculate g from QED. The nearness of $g = 2$ for the electron apparently means that the Dirac equation is an appropriate starting point. A third-rank spinor description of spin-1/2, for example, gives $g = 2/3$.

2.2 Energy-Momentum Tensor

Let us consider how fields transform under four-dimensional (space-time) coordinate transformations. Consider, first, a *scalar* field:

$$\overline{x}^\mu = x^\mu - \delta x^\mu : \quad \overline{\phi}(\overline{x}) = \phi(x), \tag{2.50}$$

That is, the value of the field at the same physical point is unchanged by the change of coordinates. Therefore,

$$\overline{\phi}(x) = \phi(x) - \delta\phi(x) = \phi(x+\delta x) = \phi(x) + \partial_\mu\phi(x)\delta x^\mu, \tag{2.51}$$

or

$$\delta\phi(x) = -\delta x^\mu \partial_\mu \phi(x). \tag{2.52}$$

The action for a free scalar field is

$$W_0 = -\frac{1}{2}\int (dx) \left(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi^2\right) = \int (dx)\,\mathcal{L}_0. \tag{2.53}$$

By varying this with respect to ϕ we recover the Klein-Gordon equation:

$$(-\partial^2 + m^2)\phi = 0. \tag{2.54}$$

Now, under the above coordinate transformation

$$\begin{aligned}
\delta\mathcal{L}_0 &= -\left(\partial^\mu\phi\partial_\mu\delta\phi + m^2\phi\,\delta\phi\right) \\
&= -\partial^\mu\phi\partial_\mu\left(-\delta x^\lambda\partial_\lambda\phi\right) - m^2\phi\left(-\delta x^\lambda\partial_\lambda\phi\right) \\
&= \delta x^\lambda\partial_\lambda\frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + m^2\phi^2\right) + (\partial_\mu\delta x^\lambda)\partial_\lambda\phi\partial^\mu\phi \\
&= -\partial_\lambda\left(\delta x^\lambda\mathcal{L}_0\right) + \partial_\mu\delta x_\nu\left(\partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\mathcal{L}_0\right).
\end{aligned} \tag{2.55}$$

For free fields, by the stationary action principle,

$$\delta W_0 = \int (dx)\delta\mathcal{L}_0 = 0, \tag{2.56}$$

so omitting surface terms as usual, we infer

$$\delta W_0 = \int (dx)\partial_\mu\delta x_\nu t_0^{\mu\nu} = 0, \tag{2.57}$$

where the energy-momentum or stress tensor is here

$$t_0^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + g^{\mu\nu}\mathcal{L}_0. \tag{2.58}$$

Note that the change in the action (2.57) would be identically zero if δx_ν were constant, i.e., for a translation. In general, the requirement of the stationary action principle (2.57) implies that this stress tensor is conserved,

$$\partial_\mu t_0^{\mu\nu} = 0. \tag{2.59}$$

This result also follows directly by using the Klein-Gordon equation. Note that $t_0^{\mu\nu} = t_0^{\nu\mu}$, that is, that the stress tensor is symmetrical in its indices.

If one integrates $t^{0\mu}$ over all space, one obtains the four-momentum:

$$P^\mu = \int (d\mathbf{x})\, t^{0\mu}, \tag{2.60}$$

and the conservation statement (2.59) says that this is a constant of the motion:

$$\dot{P}^\mu = 0. \tag{2.61}$$

Thus the conclusion is that *invariance under space-time displacements (where $\delta x^\mu =$ constant) implies the conservation of energy and momentum.*

What about a vector field, like the photon field (four-vector potential)? Since the latter possesses gauge invariance, we should be able to model the transformation properties under coordinate redefinitions by examining how a gradient of a scalar field transforms, according to (2.52):

$$\partial_\mu \delta\phi = \delta(\partial_\mu \phi) = -\delta x^\nu \partial_\nu \partial_\mu \phi - (\partial_\mu \delta x^\nu)\partial_\nu \phi. \tag{2.62}$$

We will take this to be the rule by which a vector field transforms:

$$\delta A_\mu = -\delta x^\nu \partial_\nu A_\mu - (\partial_\mu \delta x^\nu) A_\nu. \tag{2.63}$$

Let us check this by considering a Lorentz transformation (1.147), where

$$\delta x^\nu = \delta\omega^{\mu\nu} x_\mu. \tag{2.64}$$

Then (2.63) says that

$$\delta A_\mu = -\delta\omega^{\lambda\nu} x_\lambda \partial_\nu A_\mu - \delta\omega_{\mu\nu} A^\nu. \tag{2.65}$$

Indeed, under a Lorentz transformation,

$$\begin{aligned}\overline{A}^\mu(\overline{x}) = \overline{A}^\mu(x^\lambda - \delta\omega^{\alpha\lambda} x_\alpha) &= A^\mu(x) - \delta\omega^{\nu\mu} A_\nu(x) \\ &= A^\mu(x) - \delta A^\mu(x) - \partial_\lambda A^\mu \delta\omega^{\alpha\lambda} x_\alpha,\end{aligned} \tag{2.66}$$

from which we deduce

$$\delta A^\mu = \delta\omega^{\nu\mu} A_\nu - \delta\omega^{\alpha\lambda} x_\alpha \partial_\lambda A^\mu, \tag{2.67}$$

which coincides with (2.65).

Now consider the variation in the Maxwell Lagrange density, $\mathcal{L}_1 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$:

$$\begin{aligned}\delta\mathcal{L}_1 &= -\frac{1}{2}F^{\mu\nu} 2\partial_\mu \delta A_\nu = F^{\mu\nu}\partial_\mu \left[\delta x^\lambda \partial_\lambda A_\nu + (\partial_\nu \delta x^\lambda) A_\lambda\right] \\ &= -\delta x^\lambda \partial_\lambda \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) + F^{\mu\nu}\left(\partial_\mu \delta x^\lambda\right)\partial_\lambda A_\nu + F^{\mu\nu}\partial_\mu A_\lambda \partial_\nu \delta x^\lambda \\ &= -\partial_\lambda \left(-\delta x^\lambda \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) + \partial_\mu \delta x_\nu \left[g^{\mu\nu}\mathcal{L}_1 + F^{\mu\lambda}\partial^\nu A_\lambda + F^{\lambda\mu}\partial_\lambda A^\nu\right] \\ &= -\partial_\lambda(\delta x^\lambda \mathcal{L}) + (\partial_\mu \delta x_\nu) t_1^{\mu\nu},\end{aligned} \tag{2.68}$$

where the energy-momentum tensor now is

$$t_1^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda + g^{\mu\nu}\mathcal{L}_1, \tag{2.69}$$

which is again symmetric,

$$t_1^{\mu\nu} = t_1^{\nu\mu}, \tag{2.70}$$

and by the action principle, is conserved,

$$\partial_\mu t_1^{\mu\nu} = 0, \tag{2.71}$$

again entailing the conservation of energy and momentum.

It is of some significance that the Maxwell stress tensor is traceless,

$$t_1 = t_1{}^\lambda{}_\lambda = F^{\alpha\beta}F_{\alpha\beta} + 4\left(-\frac{1}{4}\right)F^{\alpha\beta}F_{\alpha\beta} = 0. \tag{2.72}$$

This reflects the *scale* invariance of the theory. A scale transformation is a particular kind of coordinate transformation

$$\delta x^\mu = \delta a\, x^\mu, \tag{2.73}$$

under which

$$\delta W = \int (dx) t_1^{\mu\nu} \partial_\mu \delta x_\nu = \int (dx) t_1^{\mu\nu} \left(\delta a\, g_{\mu\nu} + x_\nu \partial_\mu \delta a\right). \tag{2.74}$$

Given that $t = 0$, this indeed vanishes if $\delta a =$ constant, and generally, by the action principle, we have a new conserved current,

$$\partial_\mu c^\mu = 0, \quad c^\mu = x_\nu t_1^{\mu\nu}, \tag{2.75}$$

the latter being called the scale current.

Spin one-half will be treated in the homework.