We have the appropriate current already in (1.27a) and (1.27b):

$$\rho = e\psi^\dagger\psi, \quad \mathbf{j} = e\psi^\dagger\gamma^0\boldsymbol{\gamma}\psi, \tag{2.17}$$

or in four-vector notation

$$j^\mu = e\psi^\dagger\gamma^0\gamma^\mu\psi. \tag{2.18}$$

Let us verify that this indeed transforms as a four-vector:

$$j^\mu \to e\psi^\dagger\left(1 - \frac{i}{4}\sigma^{\alpha\beta\dagger}\delta\omega_{\alpha\beta}\right)\gamma^0\gamma^\mu\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\psi. \tag{2.19}$$

Here because $\gamma^{0\dagger} = \gamma^0$, $\boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}$,

$$\sigma^{ij\dagger} = \sigma^{ij}, \quad \sigma^{0i\dagger} = -\sigma^{0i}. \tag{2.20}$$

But also

$$\sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij}, \quad \sigma^{0i}\gamma^0 = -\gamma^0\sigma^{0i}, \tag{2.21}$$

so

$$\sigma^{\alpha\beta\dagger}\gamma^0 = \gamma^0\sigma^{\alpha\beta}, \tag{2.22}$$

which is to say that $\gamma^0\sigma^{\alpha\beta}$ is Hermitian. Thus, under a Lorentz transformation

$$\begin{aligned} j^\mu &\to j^\mu - \frac{ie}{4}\psi^\dagger\gamma^0[\sigma^{\alpha\beta},\gamma^\mu]\psi\delta\omega_{\alpha\beta} \\ &= j^\mu + e\psi^\dagger\gamma^0\gamma_\beta\psi\delta\omega^{\mu\beta} \\ & j^\mu + \delta\omega^{\mu\nu}j_\nu, \end{aligned} \tag{2.23}$$

which uses (1.154),

$$[\sigma^{\alpha\beta},\gamma^\mu] = 2i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha) \tag{2.24}$$

2.1 Action Principles

The conservation of electric charge, like all conservation laws, reflects an underlying symmetry. To discuss this, we need the Dirac and Maxwell *actions*. Define the functionals

$$W_D[\overline{\psi},\psi] = -\int(dx)\overline{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\psi, \tag{2.25a}$$

$$W_M[A] = \int(dx)\left(-\frac{1}{4}\right)F^{\mu\nu}F_{\mu\nu}, \tag{2.25b}$$

where integration is assumed over all space-time, with measure $(dx) = dt\,(d\mathbf{x})$, and where

$$\overline{\psi} = \psi^\dagger\gamma^0, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{2.26}$$

We regard $\overline{\psi}$, ψ, and A as *independent* variables, and vary the action with respect to each in turn:

$$\begin{aligned}\delta W_D &= -\int (dx)\left[\delta\overline{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\psi + \overline{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\delta\psi\right]\\ &= -\int (dx)\left[\delta\overline{\psi}\left(\gamma\frac{1}{i}\partial + m\right)\psi + \overline{\psi}\left(-\gamma\frac{1}{i}\overleftarrow{\partial} + m\right)\delta\psi\right], \qquad (2.27)\end{aligned}$$

where, in the last, we have integrated by parts, and *omitted surface terms*. The notation means

$$\overline{\psi}\overleftarrow{\partial}_\mu = \partial_\mu\overline{\psi}. \qquad (2.28)$$

Similarly,

$$\delta W_M = -\int (dx) F^{\mu\nu}\partial_\mu\delta A_\nu = \int (dx)\partial_\mu F^{\mu\nu}\delta A_\nu. \qquad (2.29)$$

Now we impose the *Stationary Action Principle*: $\delta W = 0$. This is to say that the "true path" is an extremum. The equations of motion follow:

$$\left(\gamma\frac{1}{i}\partial + m\right)\psi = 0, \quad \overline{\psi}\left(-\gamma\frac{1}{i}\overleftarrow{\partial} + m\right) = 0, \qquad (2.30a)$$

$$\partial_\mu F^{\mu\nu} = 0. \qquad (2.30b)$$

These are the free Dirac and Maxwell equations. Note that the two forms of the Dirac equation are equivalent because $\gamma^0\gamma^\mu$ is Hermitian.

Now the Maxwell action is invariant under the gauge transformations,

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad (2.31)$$

while the Dirac action is invariant under the global (constant) phase transformation

$$\psi \to e^{i\Lambda}\psi, \quad \overline{\psi} \to e^{-i\Lambda}\overline{\psi}, \qquad (2.32)$$

where Λ is a constant number. What if Λ is not constant? Then the Dirac action changes:

$$\begin{aligned}\delta W_D &= -\int (dx)\,\overline{\psi}e^{-i\Lambda}\gamma^\mu\frac{1}{i}(\partial_\mu e^{i\Lambda})\psi\\ &= -\int (dx)\,\partial_\mu\Lambda\overline{\psi}\gamma^\mu\psi. \qquad (2.33)\end{aligned}$$

Since δW_D is stationary for arbitrary infinitesimal variations about solutions of the equations of motion, we conclude that, because Λ is an arbitrary function,

$$\partial_\mu\overline{\psi}\gamma^\mu\psi = 0, \qquad (2.34)$$

which of course can be directly proved from the equations of motion, as we did in Chapter 1, in Eq. (1.26). Equation (2.34) is just the statement of (electric) current conservation (2.15).

But let us go one step further and require that the action be *identically* invariant under local $\Lambda(x)$ transformations. We will need to add some term $W_{\rm int}$ to W_D and W_M such that

$$\delta W_{\rm int} = +\int (dx)\, \partial_\mu \Lambda \overline{\psi}\gamma^\mu \psi. \tag{2.35}$$

This will be the case if

$$W_{\rm int} = e\int (dx)\, A_\mu \overline{\psi}\gamma^\mu \psi, \tag{2.36}$$

where $\delta A_\mu = \partial_\mu \lambda$, and we have identified the gauge and phase transformation parameters,

$$\lambda = \frac{1}{e}\Lambda, \tag{2.37}$$

where e is the arbitrary charge of the electron. We have united two independent symmetries, gauge transformations on Maxwell fields and global phase transformations on Dirac fields, which apply for free fields, into a single local gauge transformation symmetry of interacting electron and photon fields:

$$A_\mu \;\to\; A_\mu + \partial_\mu \lambda, \tag{2.38a}$$
$$\psi \;\to\; e^{ie\lambda}\psi, \quad \overline{\psi} \to \overline{\psi}e^{-ie\lambda}. \tag{2.38b}$$

The action which is invariant under these transformations is

$$W = -\int (dx) \left\{ \overline{\psi}\left[\gamma^\mu \left(\frac{1}{i}\partial_\mu - eA_\mu\right) + m\right]\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}. \tag{2.39}$$

We see the appearance of the gauge-covariant derivative,

$$D_\mu = \partial_\mu - ieA_\mu, \tag{2.40a}$$

which we had seen earlier as the gauge-covariant momentum operator (1.31)

$$\pi_\mu = \frac{1}{i}\partial_\mu - eA_\mu. \tag{2.40b}$$

π_μ satisfies the commutation relation

$$[\pi_\mu, \pi_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ieF_{\mu\nu}. \tag{2.41}$$

By gauge covariant, we mean, under a gauge transformation,

$$D_\mu \to e^{-ie\lambda}(D_\mu - ie\partial_\mu \lambda)e^{ie\lambda} = D_\mu. \tag{2.42}$$

The resulting Dirac and Maxwell equations are

$$(\gamma\pi + m)\psi \;=\; 0, \quad \pi = p - eA, \tag{2.43a}$$
$$\partial_\nu F^{\mu\nu} \;=\; j^\mu, \quad j^\mu = e\overline{\psi}\gamma^\mu\psi, \tag{2.43b}$$

just as advertized.

This "gauging" procedure leads to the *minimal* interaction between electron and photon, but there can be further interactions, as long as they are gauge invariant. The *anomalous magnetic moment* term

$$W' = -\frac{\kappa}{2}\int (dx)\overline{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi \tag{2.44}$$

is gauge invariant by itself, because F appears and no derivatives act on ψ. This adds to the Dirac equation the term

$$\frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}\psi = \kappa\mathbf{\Sigma}\cdot\mathbf{H}\psi \tag{2.45}$$

for a pure magnetic field. If such an extra term were present, the net magnetic moment would be [see (1.39)

$$\boldsymbol{\mu} = \left(\frac{e}{2m}+\kappa\right)\mathbf{\Sigma} = \frac{e}{2m}\frac{g}{2}\mathbf{\Sigma}, \tag{2.46}$$

which is to say

$$\kappa = \frac{e}{2m}\left(\frac{g-2}{2}\right). \tag{2.47}$$

However, experimentally, for the electron, g is very nearly 2:

$$\frac{g_e}{2} = 1.001159652187(4), \tag{2.48}$$

and similarly for the muon:

$$\frac{g_\mu}{2} = 1.0011659160(6). \tag{2.49}$$

Does this mean that a small κ value should be inserted ad hoc? No! We will calculate g from QED. The nearness of $g = 2$ for the electron apparently means that the Dirac equation is an appropriate starting point. A third-rank spinor description of spin-1/2, for example, gives $g = 2/3$.