Chapter 2

Abelian Gauge Fields

Maxwell's equations in 3-vector notation are (Heaviside-Lorentz units with $c = 1$)

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho, \quad \boldsymbol{\nabla}\times\mathbf{B} - \dot{\mathbf{E}} = \mathbf{j}, \tag{2.1a}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \dot{\mathbf{B}} = 0, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{2.1b}$$

The last two equations are identically satisfied if $\mathbf{E}$ and $\mathbf{B}$ are constructed from a scalar and vector potential

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \dot{\mathbf{A}}, \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{2.2}$$

The potentials ϕ and $\mathbf{A}$ are not uniquely defined, because they may be redefined by a *gauge transformation*:

$$\phi \to \phi - \dot{\lambda}, \quad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda, \tag{2.3}$$

where $\lambda(\mathbf{r}, t)$ is an arbitrary function of space and time. The electric and magnetic fields, and correspondingly energy, momentum density, and the like, are *gauge invariant*:

$$\mathbf{E} \to \mathbf{E}, \quad \mathbf{B} \to \mathbf{B}. \tag{2.4}$$

The potentials are therefore not physically observable.

As Poincaré and Einstein noted, Maxwell's equations transform covariantly under Lorentz transformations. This is seen most transparently if they are written in four-tensor notation. Define an antisymmetric Maxwell field strength tensor $F^{\mu\nu}$ by

$$F^{ij} = \epsilon^{ijk} B_k. \quad F^{0i} = E^i, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{2.5}$$

We further define the electric current density by

$$j^\mu = (\rho, \mathbf{j}). \tag{2.6}$$

Then it is easy to check that Maxwell's equations read

$$\partial_\nu F^{\mu\nu} = j^\mu, \tag{2.7a}$$

$$\partial_\nu {}^*F^{\mu\nu} = 0, \tag{2.7b}$$

where the dual tensor ${}^*F^{\mu\nu}$ is defined by

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}. \tag{2.8}$$

Thus,

$${}^*F^{ij} = -\epsilon^{ijk}E_k, \quad {}^*F^{0i} = B^i, \tag{2.9}$$

which exhibits the dual operation:

$$*: \quad \mathbf{E} \to \mathbf{B}, \quad \mathbf{B} \to -\mathbf{E}. \tag{2.10}$$

Duality would be a symmetry of Maxwell's equations if there were magnetic charge. See homework. Note further that conservation of electric charge

$$\partial_\mu j^\mu = \frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{2.11}$$

is a consequence of the first Maxwell equation (2.7a), because $\partial_\mu\partial_\nu F^{\mu\nu} = 0$.

The dual equation is an identity (Bianchi identity) if F is constructed from a vector potential a:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad A^\mu = (\phi, \mathbf{A}). \tag{2.12}$$

The Lorentz covariance of Maxwell's equations follows when it is demonstrated that A^μ, j^μ transform as four-vectors, i.e., as $(1/2, 1/2)$. We'll take this for granted:

$$A^\mu \to A^\mu - \delta\omega^{\nu\mu}A_\nu, \tag{2.13a}$$

$$j^\mu \to j^\mu - \delta\omega^{\nu\mu}j_\nu. \tag{2.13b}$$

In four-vector language, a gauge transformation is

$$A^\mu \to A^\mu + \partial^\mu\lambda. \tag{2.14}$$

Now, how does the electron interact with the electromagnetic field (photons)? It interacts through the current j^μ. This current must be conserved,

$$\partial_\mu j^\mu = 0, \tag{2.15}$$

which implies that total charge is conserved:

$$\frac{d}{dt}\int (d\mathbf{x})\rho(\mathbf{x}, t) = \dot{Q} = 0. \tag{2.16}$$

We have the appropriate current already in (1.27a) and (1.27b):

$$\rho = e\psi^\dagger\psi, \quad \mathbf{j} = e\psi^\dagger\gamma^0\boldsymbol{\gamma}\psi, \tag{2.17}$$

or in four-vector notation

$$j^\mu = e\psi^\dagger\gamma^0\gamma^\mu\psi. \tag{2.18}$$

Let us verify that this indeed transforms as a four-vector:

$$j^\mu \to e\psi^\dagger\left(1 - \frac{i}{4}\sigma^{\alpha\beta\dagger}\delta\omega_{\alpha\beta}\right)\gamma^0\gamma^\mu\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\psi. \tag{2.19}$$

Here because $\gamma^{0\dagger} = \gamma^0$, $\boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}$,

$$\sigma^{ij\dagger} = \sigma^{ij}, \quad \sigma^{0i\dagger} = -\sigma^{0i}. \tag{2.20}$$

But also

$$\sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij}, \quad \sigma^{0i}\gamma^0 = -\gamma^0\sigma^{0i}, \tag{2.21}$$

so

$$\sigma^{\alpha\beta\dagger}\gamma^0 = \gamma^0\sigma^{\alpha\beta}, \tag{2.22}$$

which is to say that $\gamma^0\sigma^{\alpha\beta}$ is Hermitian. Thus, under a Lorentz transformation

$$\begin{aligned} j^\mu \to\ & j^\mu - \frac{ie}{4}\psi^\dagger\gamma^0[\sigma^{\alpha\beta}, \gamma^\mu]\psi\delta\omega_{\alpha\beta} \\ =\ & j^\mu + e\psi^\dagger\gamma^0\gamma_\beta\psi\delta\omega^{\mu\beta} \\ & j^\mu + \delta\omega^{\mu\nu}j_\nu, \end{aligned} \tag{2.23}$$

which uses (1.154),

$$[\sigma^{\alpha\beta}, \gamma^\mu] = 2i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha) \tag{2.24}$$