Chapter 1

Dirac Equation

This course will be devoted principally to an exposition of the dynamics of Abelian and non-Abelian gauge theories. These form the basis of the *Standard Model*, that is, the theory of (partially unified) electromagnetic and weak interactions, and of the strong interaction described by quantum chromodynamics (QCD). This model fundamentally describes leptons and quarks interacting via gauge bosons, where the masses of the particles are generated, presumably, by the Higgs mechanism. Our paradigm of such a theory of particles and forces, a *gauge theory*, is quantum electrodynamics (QED). QED is the theory of charged leptons (e, μ, τ) interacting with and through the photon γ.

To begin this process, we must study relativistic spin-1/2 particles, which include leptons and quarks. The relativistic wave equation governing such particles is the *Dirac equation*, which we motivate as follows. (Throughout this course we set $\hbar = c = 1$.)

Nonrelativistic quantum mechanics is governed by the Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi = H\psi, \tag{1.1}$$

where the Hamiltonian for a *free* particle is

$$H = \frac{p^2}{2m} = -\frac{1}{2m}\nabla^2. \tag{1.2}$$

This dynamical equation is evidently not compatible with Einstein's relativity because space and time are treated on different footings. A hint of the correct way to proceed is given by the relativistic equation relating energy and momentum,

$$E^2 = p^2 + m^2; \tag{1.3}$$

if we make the operator replacements

$$E \to i\frac{\partial}{\partial t}, \quad \mathbf{p} \to \frac{1}{i}\boldsymbol{\nabla}, \tag{1.4}$$

we obtain a relativistic wave equation

$$-\frac{\partial^2}{\partial t^2}\psi = (-\nabla^2 + m^2)\psi. \tag{1.5}$$

This may be covariantly expressed using 4-vector notation

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right). \tag{1.6}$$

The corresponding contravariant derivative is given by

$$\frac{\partial}{\partial x_\mu} = \partial^\mu = g^{\mu\nu}\partial_\nu = \left(-\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \tag{1.7}$$

where we are using the "democratic" (i.e., majority rules) metric,

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.8}$$

The wave operator or the d'Alembertian is given by the square of the 4-derivative,

$$\partial^2 = \partial^\mu\partial_\mu = -\partial_0^2 + \nabla^2. \tag{1.9}$$

Thus the wave equation (1.5) has the covariant form

$$(-\partial^2 + m^2)\psi = 0. \tag{1.10}$$

This is called the Klein-Gordon equation.

However, the fact that the Klein-Gordon equation is second order in time means that we cannot attach a probability interpretation to ψ. The conserved current is

$$j^\mu = \psi^* \frac{1}{i}\partial^\mu\psi - \psi\frac{1}{i}\partial^\mu\psi^*, \tag{1.11}$$

for

$$i\partial_\mu j^\mu = \psi^*\partial^2\psi - \psi\partial^2\psi^* = m^2(\psi^*\psi - \psi\psi^*) = 0. \tag{1.12}$$

But the quantity we would want to interpret as the probability density,

$$\rho = j_0 = \psi^*\frac{1}{i}\frac{\partial}{\partial t}\psi - \psi\frac{1}{i}\frac{\partial}{\partial t}\psi^*, \tag{1.13}$$

is *not* positive definite.

This suggests that we should try to construct a *first-order* equation by taking the "square-root" of the Klein-Gordon equation. Let us assume a form

$$i\frac{\partial}{\partial t}\psi = \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m\right)\psi, \tag{1.14}$$

and see what the quantities $\boldsymbol{\alpha}$ and β have to be in order for this equation to be relativistically covariant. Let us iterate this equation:

$$\begin{aligned}
-\frac{\partial^2}{\partial t^2}\psi &= i\frac{\partial}{\partial t}\left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m\right)\psi \\
&= \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m\right)\left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m\right)\psi \\
&= (-\nabla^2+m^2)\psi, \qquad (1.15)
\end{aligned}$$

provided

$$\begin{aligned}
\frac{1}{2}(\alpha_i\alpha_j+\alpha_j\alpha_i) &= \frac{1}{2}\{\alpha_i,\alpha_j\} = \delta_{ij}, \qquad (1.16a)\\
\alpha_i\beta+\beta\alpha_i &= 0, \qquad (1.16b)\\
\beta^2 &= 1. \qquad (1.16c)
\end{aligned}$$

This cannot be true if $\boldsymbol{\alpha}$, β are numbers, but could be satisfied if they are matrices. Correspondingly, the ψ must be a column vector. What else can we say about the matrices $\boldsymbol{\alpha}$ and β? Since $H=\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m$ must be Hermitian, so must $\boldsymbol{\alpha}$ and β:

$$\boldsymbol{\alpha}^\dagger=\boldsymbol{\alpha}, \quad \beta^\dagger=\beta. \qquad (1.17)$$

From $\alpha_i^2=\beta^2=1$ we conclude that the eigenvalues of α_i and β are ±1. Further,

$$\alpha_i=\alpha_i\beta^2=-\beta\alpha_i\beta, \qquad (1.18)$$

and so

$$\mathrm{Tr}\,\alpha_i=-\mathrm{Tr}\,\beta\alpha_i\beta=-\mathrm{Tr}\,\alpha_i\beta^2=-\mathrm{Tr}\,\alpha_i=0. \qquad (1.19)$$

Similarly, (no sum on i)

$$\mathrm{Tr}\,\beta=\mathrm{Tr}\,\beta\alpha_i^2=-\mathrm{Tr}\,\alpha_i\beta\alpha_i=-\mathrm{Tr}\,\beta=0. \qquad (1.20)$$

These results mean that the dimensions of the matrices must be an even number, since the number of positive eigenvalues must equal the number of negative eigenvalues. It cannot be 2, because there are only *three* 2×2 matrices, σ_x, σ_y, and σ_z, beside 1. The smallest representation of these Dirac matrices is thus 4×4. There are many different solutions to the above equations. In this course, I will restrict myself to the class of *Majorana representations*, which satisfy the additional subsidiary conditions: β is antisymmetric and imaginary:

$$\beta^*=-\beta, \quad \beta^T=-\beta, \qquad (1.21a)$$

while the α_i are symmetric and real:

$$\boldsymbol{\alpha}^*=\boldsymbol{\alpha}, \quad \boldsymbol{\alpha}^T=\boldsymbol{\alpha}. \qquad (1.21b)$$

In fact, it is almost never necessary to use an explicit representation, but here is an example of one anyway:

$$\beta = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} = \tau_3 \otimes \sigma_2, \tag{1.22a}$$

$$\alpha_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = 1 \otimes \sigma_1, \tag{1.22b}$$

$$\alpha_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \tau_2 \otimes \sigma_2, \tag{1.22c}$$

$$\alpha_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = 1 \otimes \sigma_3, \tag{1.22d}$$

which are expressed in 2×2 block form in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.23}$$

and similarly for τ_i. I will leave the verification that these satisfy all the requirements to the homework.

Now, what about the probability current associated with the Dirac equation (1.14)? Multiply (1.14) on the left by $\psi^\dagger$, and the adjoint of the Dirac equation on the right by ψ and subtract:

$$\psi^\dagger i\frac{\partial}{\partial t}\psi = \psi^\dagger \left(\frac{1}{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m\right)\psi, \tag{1.24a}$$

$$\left(i\frac{\partial}{\partial t}\psi^\dagger\right)\psi = -\left(-\frac{1}{i}\boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha} + m\psi^\dagger\beta\right)\psi, \tag{1.24b}$$

resulting in

$$i\frac{\partial}{\partial t}\left(\psi^\dagger\psi\right) = \frac{1}{i}\boldsymbol{\nabla}\cdot\left(\psi^\dagger\boldsymbol{\alpha}\psi\right), \tag{1.25}$$

or

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\mathbf{j} = 0, \tag{1.26}$$

which is the continuity equation, apparently expressing the conservation of probability, where the probability density is

$$\rho = \psi^\dagger\psi > 0, \tag{1.27a}$$

and the probability current is

$$\mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi. \tag{1.27b}$$

The consequence is the constancy of the total probability of finding the electron at any point is space,

$$\frac{d}{dt}\int(d\mathbf{r})\psi^\dagger\psi = 0. \tag{1.28}$$

Thus we have apparently solved the problem of finding a relativistic generalization of the Schrödinger equation.

1.1 Nonrelativistic Limit

Suppose we put the particle in a magnetic field $\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{A}$. Then, anticipating the minimal substitution, the Dirac equation becomes ($\mathbf{p} = (1/i)\boldsymbol{\nabla}$)

$$i\frac{\partial}{\partial t}\psi = [\boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m]\,\psi, \tag{1.29}$$

which, when iterated, becomes

$$-\frac{\partial^2}{\partial t^2}\psi = \left[\alpha_i\alpha_j\pi_i\pi_j + m^2\right]\psi, \tag{1.30}$$

where the gauge-covariant momentum has been introduced,

$$\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}. \tag{1.31}$$

The product of Dirac matrices appearing here can be written in terms of symmetric and antisymmetric parts,

$$\alpha_i\alpha_j = \frac{1}{2}\{\alpha_i, \alpha_j\} + \frac{1}{2}[\alpha_i, \alpha_j] = \delta_{ij} + i\epsilon_{ijk}\Sigma_j, \tag{1.32}$$

where we have introduced the totally antisymmetric symbol ϵ_{ijk}, which vanishes if any two of its indices are equal, equalling 1 if ijk is an even permutation of 123, and -1 if it is an odd permutation. $\boldsymbol{\Sigma}$ is called the spin matrix. Thus the iterated Dirac equation equals

$$-\frac{\partial^2}{\partial t^2}\psi = \left[m^2 + (\mathbf{p} - e\mathbf{A})^2 + i\boldsymbol{\Sigma} \cdot \boldsymbol{\pi} \times \boldsymbol{\pi}\right]\psi. \tag{1.33}$$

But

$$\boldsymbol{\pi} \times \boldsymbol{\pi} = \left(\frac{1}{i}\boldsymbol{\nabla} - e\mathbf{A}\right) \times \left(\frac{1}{i}\boldsymbol{\nabla} - e\mathbf{A}\right) = -\frac{1}{i}e\boldsymbol{\nabla} \times \mathbf{A} = ie\mathbf{H}, \tag{1.34}$$

so we recover a modified Klein-Gordon equation,

$$-\frac{\partial^2}{\partial t^2}\psi = \left[(\mathbf{p} - e\mathbf{A})^2 + m^2 - e\boldsymbol{\Sigma} \cdot \mathbf{H}\right]\psi. \tag{1.35}$$

The operator $\boldsymbol{\Sigma}$ indeed refers to spin one-half, because

$$\Sigma_i\Sigma_j = \delta_{ij} + i\epsilon_{ijk}\Sigma_k, \tag{1.36}$$

which again you will verify in homework. This is the same equation the Pauli matrices satisfy. Like the Pauli matrices, the Σ_i have eigenvalues ± 1.

We identify the nonrelativistic energy W by

$$-\frac{\partial^2}{\partial t^2} \to E^2 = (m + W)^2 \approx m^2 + 2mW, \quad m \gg W, \tag{1.37}$$

so

$$W = \frac{(\mathbf{p} - e\mathbf{A})}{2m} - \frac{e}{2m}\mathbf{\Sigma} \cdot \mathbf{H}, \tag{1.38}$$

where we see the appearance of the magnetic moment of the electron

$$\boldsymbol{\mu} = \frac{e}{2m}\mathbf{\Sigma}. \tag{1.39}$$

Here we recognize the Bohr magneton, the implication being that the g-factor of the electron is 2, which is very nearly true.

1.2 Rotational Invariance of the Dirac Equation

The above discussion suggests that the particle described by the Dirac equation has intrinsic angular momentum $\frac{1}{2}\hbar$, so that the total angular momentum is

$$\mathbf{J} = \mathbf{L} + \frac{1}{2}\mathbf{\Sigma}, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1.40}$$

If so, the Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \tag{1.41}$$

should commute with the total angular momentum,

$$[\mathbf{J}, H] = 0. \tag{1.42}$$

The orbital angular momentum commutation relation is

$$[L_i, H] = [\epsilon_{ijk} x_j p_k, \alpha_l p_l] = \alpha_l \epsilon_{ijk} i \delta_{jl} p_k = i(\boldsymbol{\alpha} \times \mathbf{p})_i, \tag{1.43}$$

or

$$[\mathbf{L}, H] = i\boldsymbol{\alpha} \times \mathbf{p}. \tag{1.44}$$

On the other hand, from the definition

$$\frac{1}{2}[\alpha_i, \alpha_j] = i\epsilon_{ijk}\Sigma_k, \tag{1.45}$$

we obtain

$$\epsilon_{ijk}\frac{1}{2}[\alpha_i, \alpha_j] = \epsilon_{ijk}\alpha_i\alpha_j = i\epsilon_{ijk}\epsilon_{ijl}\Sigma_l = 2i\Sigma_k, \tag{1.46}$$

or

$$\mathbf{\Sigma} = \frac{1}{2i}\boldsymbol{\alpha} \times \boldsymbol{\alpha}. \tag{1.47}$$

From this it is a simple matter to work out the commutator of the spin operator with the Hamiltonian:

$$\begin{aligned}
\left[\frac{1}{2}\Sigma_i, H\right] &= \frac{1}{4i}\left[\epsilon_{ijk}\alpha_j\alpha_k, \alpha_l\right] p_l \\
&= \frac{1}{4i} p_l \epsilon_{ijk} \left(\alpha_j\{\alpha_k, \alpha_l\} - \{\alpha_j, \alpha_l\}\alpha_k\right) \\
&= \frac{1}{2i} p_l (\epsilon_{ijl}\alpha_j - \epsilon_{ilk}\alpha_k) \\
&= \frac{1}{i}\epsilon_{ijl}\alpha_j p_l = -i(\boldsymbol{\alpha} \times \mathbf{p})_i.
\end{aligned} \tag{1.48}$$

Thus the required rotational invariance statement is verified:

$$[\mathbf{J}, H] = [\mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}, H] = i\boldsymbol{\alpha} \times \mathbf{p} - i\boldsymbol{\alpha} \times \mathbf{p} = 0. \tag{1.49}$$

1.3 Translational Invariance

One of the invariances of any isolated physical system is the freedom to change the origin of time. Let us imagine a change in the time coordinate,

$$t \to \bar{t} = t - \delta t, \quad \text{where } \delta t \text{ is constant.} \tag{1.50}$$

In going from t to $\bar{t}$, the origin of time is shifted *forward* by an amount δt. Under such a change, states and operators do not change. However, we want to introduce *new* states and operators which have the same properties relative to the new time coordinate $\bar{t}$ as the old states and operators had relative to the old time coordinate t:

$$t \text{ coordinate}: \quad X, \quad |\,\rangle, \quad \langle\,|, \tag{1.51a}$$

$$\bar{t} \text{ coordinate}: \quad \overline{X}, \quad \overline{|\,\rangle}, \quad \overline{\langle\,|}. \tag{1.51b}$$

The new states and operators have the same inter-relations as the old states and operators; therefore, the two sets are related by a *unitary transformation*:

$$\overline{X} = U^{-1}XU, \quad U^{-1} = U^\dagger, \tag{1.52a}$$

$$\overline{\langle\,|} = \langle\,|U, \quad \overline{|\,\rangle} = U^\dagger|\,\rangle. \tag{1.52b}$$

What can we say about the unitary operator U? If $\delta t = 0$, the change in the states and operators is zero, so $U = 1$. If $\delta t \neq 0$, but very small, U must differ infinitesimally from 1. We therefore write

$$U = 1 - \frac{i}{\hbar}\delta t\, H. \tag{1.53}$$

We'll see in a moment why it's convenient to have the $-i/\hbar$ factor. What are the properties of H? U must be unitary so

$$1 = U^\dagger U = \left(1 + \frac{i}{\hbar}\delta t\, H^\dagger\right)\left(1 - \frac{i}{\hbar}\delta t\, H\right) = 1 - \frac{i}{\hbar}(H - H^\dagger)\delta t + O(\delta t^2). \tag{1.54}$$

Therefore H must be Hermitian, $H = H^\dagger$, which is why the i was put in front. We conclude that H is a physical quantity. It corresponds to the energy of the system; we call it the energy operator or the *Hamiltonian*. It has the right dimensions to be an energy, since $[\hbar] = \text{energy} \times \text{time}$. The Hamiltonian is the generator of time translations.

A *dynamical variable* is an operator characterizing in part a dynamical system, which changes as time evolves. An example we've discussed so far is the