9-1 Homework

Under a gauge transformation

$$F_{\mu\nu} \to U F_{\mu\nu} U^+ \equiv F^U_{\mu\nu}$$

How does this look for the components, defined by $F_{\mu\nu} = F^a_{\mu\nu} \frac{\tau^a}{2}$?

$$F^a_{\mu\nu} \to (F^U_{\mu\nu})^a = Tr\, \tau^a F^U_{\mu\nu}$$

$$= Tr\, \tau^a U F^b_{\mu\nu} \frac{\tau^b}{2} U^+$$

$$= F^b_{\mu\nu} \frac{1}{2} Tr(\tau^a U \tau^b U^+)$$

Then, $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$

$$\to -\frac{1}{4} F^b_{\mu\nu} F^{c,\mu\nu} \frac{1}{4} Tr(\tau^a U \tau^b U^+) \times Tr(\tau^a U \tau^c U^+)$$

However $\sum_{a=1}^{3} \tau^a_{ij} \tau^a_{kl} = 2(\delta_{il}\delta_{kj} - \frac{1}{2}\delta_{ij}\delta_{kl})$

as proved below, so

$$(\mathrm{Tr}\,\tau^a U \tau^b U^+)(\mathrm{Tr}\,\tau^a U \tau^c U^+)$$

$$= (\tau^a_{ij} U_{jk} \tau^b_{kl} U^+_{li})(\tau^a_{rs} U_{st} \tau^c_{tu} U^+_{ur})$$

$$= 2(\delta_{is}\delta_{rj} - \tfrac{1}{2}\delta_{ij}\delta_{rs})\, U_{jk} \tau^b_{kl} U^+_{li} U_{st} \tau^c_{tu} U^+_{ur}$$

$$= 2\,\mathrm{Tr}(U\tau^b U^+ U \tau^c U^+) - \mathrm{Tr}(U\tau^b U^+)\,\mathrm{Tr}(U\tau^c U^+)$$

$$= 2\,\mathrm{Tr}\,\tau^b\tau^c - (\mathrm{Tr}\,\tau^b)(\mathrm{Tr}\,\tau^c) \quad \text{using props of trace}$$

But $\mathrm{Tr}\,\tau^a = 0$, and $\tau^b\tau^c = \delta_{bc} + i\varepsilon_{bca}\tau^a$

Above trace $= 2\cdot 2\,\delta_{bc} = 4\delta_{bc}$

and then $\mathcal{L} \to \mathcal{L}$, QED.

<u>Proof of lemma</u> $\tau^a_{ij}\tau^a_{kl} = 2(\delta_{il}\delta_{kj} - \tfrac{1}{2}\delta_{ij}\delta_{kl})$

The only tensor available is δ_{ij}, so we must have

$$\tau^a_{ij}\tau^a_{kl} = A\,\delta_{ij}\delta_{kl} + B\,\delta_{ik}\delta_{jl} + C\,\delta_{il}\delta_{jk}$$

where A, B, and C are constant. To determine these: 1) Set $j = k$ and sum:

$$(\tau^a\tau^a)_{il} = A\,\delta_{il} + B\delta_{il} + 2C\,\delta_{il}$$

$$(\delta_{kk} = 2)$$

But $\iota^{\wedge}\tau^a = 3(\mathbb{1})$, $(\tau^a\tau^a)_{il} = 3\delta_{il}$

$$\Rightarrow \quad 3 = A + B + 2C \qquad (1)$$

2) Set $i=j$ and sum:

$$0 = (\mathrm{Tr}\,\tau^a)\,\tau^a_{kl} = 2A\delta_{kl} + B\delta_{kl} + C\delta_{kl}$$

$$0 = 2A + B + C \qquad (2)$$

3) Set $i=k$ and sum:

$$\tau^a_{ij}\tau^a_{il} = A\delta_{jl} + 2B\delta_{jl} + C\delta_{jl}$$

$$\tau^a_{ij}\tau^a_{il} = (\tau^{aT}\tau^a)_{jl}$$

$$= \left[(\tau^1)^2 - (\tau^2)^2 + (\tau^3)^2\right]_{jl} = [\mathbb{1} - \mathbb{1} + \mathbb{1}]_{jl}$$

$$= \delta_{jl}$$

$$\Rightarrow \quad 1 = A + 2B + C \qquad (3)$$

$(3)-(1) \Rightarrow 1 = B - A$; $(1) - 2(2) \Rightarrow 3 = -B - 3A$

$$\Rightarrow A = -1, \; B = 0, \; C = 2 \qquad \text{QED.}$$

9-2 If we start from

$$\mathcal{D}_\mu \omega = \partial_\mu \omega + ig[A_\mu, \omega] = [\partial_\mu + igA_\mu, \omega]$$

for the definition of the covariant derivative for an adjoint (spin-1) field, $\omega = \omega^a \tau^a/2$, then

$$(\mathcal{D}_\mu \omega)^a \frac{\tau^a}{2} = (\partial_\mu \omega^a)\frac{\tau^a}{2} + ig\underbrace{\left[A_\mu^b \frac{\tau^b}{2}, \omega^c \frac{\tau^c}{2}\right]}_{A_\mu^b \omega^c i\varepsilon_{abc}\frac{\tau^a}{2}}$$

$$(\mathcal{D}_\mu \omega)^a = \partial_\mu \omega^a + g\varepsilon_{abc}\,\omega^b A_\mu^c$$

$$\parallel$$

$$(\mathcal{D}_\mu)^{ab}\omega^b$$

or $(\mathcal{D}_\mu)_{ab} = \delta_{ab}\partial_\mu + g\varepsilon_{abc}A_\mu^c$

Now we must show $\mathcal{D}_\mu$ transforms covariantly under a gauge transformation, that is

$$\mathcal{D}_\mu \to U\mathcal{D}_\mu U^+.$$

[Obvious, since $\partial_\mu + igA_\mu \to U(\partial_\mu + igA_\mu)U$]

We do this by noting that under a gauge transformation, $\omega \to U\omega U^+$ and

$$\mathcal{D}_\mu \omega \to \partial_\mu (U\omega U^+) + ig\left[UA_\mu U^+ - \frac{i}{g} U\partial_\mu U^+, U\omega U^+\right]$$

$$= U(\partial_\mu \omega)U^+ + \cancel{(\partial_\mu U)\omega U^+} + \cancel{U\omega(\partial_\mu U^+)}$$
$$+ igU[A_\mu, \omega]U^+ + \underbrace{[U\partial_\mu U^+, U\omega U^+]}_{\cancel{U\partial_\mu U^+}U\omega U^+ - \cancel{U\omega\partial_\mu} U^+}$$

since $U\partial_\mu U^+ + (\partial_\mu U)U^+ = 0$

$$\therefore \mathcal{D}_\mu \omega \to U(\mathcal{D}_\mu \omega)U^+ = U\mathcal{D}_\mu U^+ U\omega U^+$$

$$\Rightarrow \mathcal{D}_\mu \to U\mathcal{D}_\mu U^+ \qquad \text{QED.}$$

9-3. $\lambda_1 \lambda_1 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} = a\mathbb{1} + b\lambda_8$

$$1 = a + \frac{b}{\sqrt{3}}, \quad 0 = a - \frac{2}{\sqrt{3}} b,$$

$$a = \frac{2}{\sqrt{3}} b, \quad 1 = \frac{3}{\sqrt{3}} b, \quad b = \frac{1}{\sqrt{3}}, \quad a = \frac{2}{3}$$

$$d_{118} = \frac{1}{\sqrt{3}}$$

$$\lambda_2 \lambda_2 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \Rightarrow d_{228} = \frac{1}{\sqrt{3}}$$

$$\lambda_3 \lambda_3 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \Rightarrow d_{338} = \frac{1}{\sqrt{3}}$$

$$\lambda_4 \lambda_4 = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 1 \end{pmatrix} = a\mathbb{1} + b\lambda_8 + c\lambda_3$$

$$\left.\begin{aligned} 1 &= a + \frac{b}{\sqrt{3}} + c \\ 0 &= a + \frac{b}{\sqrt{3}} - c \end{aligned}\right\} \quad 2c = 1, \quad c = \frac{1}{2}$$

$$\left.\begin{aligned} 1 &= a - \frac{2b}{\sqrt{3}} \\ \text{first 2 eqn} \Rightarrow \quad 1 &= 2a + \frac{2b}{\sqrt{3}} \end{aligned}\right\} \quad 3a = 2, \quad a = \frac{2}{3}$$

$$b = \sqrt{3}\left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{2\sqrt{3}}$$

$$d_{448} = -\frac{1}{2\sqrt{3}}, \quad d_{443} = \frac{1}{2}$$

$$\lambda_5\lambda_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow d_{558} = -\frac{1}{2\sqrt{3}}, \; d_{553} = \frac{1}{2}$$

$$\lambda_6\lambda_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a\mathbb{1} + b\lambda_8 + c\lambda_3$$

$$\left.\begin{array}{l} 0 = a + \frac{b}{\sqrt{3}} + c \\ 1 = a + \frac{b}{\sqrt{3}} - c \end{array}\right\} \quad 1 = 2a + \frac{2b}{\sqrt{3}}$$

$$1 = a - \frac{2b}{\sqrt{3}}$$

$$2 = 3a, \quad a = \frac{2}{3} \checkmark$$

$$b = \frac{\sqrt{3}}{2}\left(\frac{2}{3} - 1\right) = -\frac{1}{2\sqrt{3}}$$

$$c = -\frac{2}{3} + \frac{1}{6} = -\frac{1}{2}$$

$$d_{668} = -\frac{1}{2\sqrt{3}}, \quad d_{663} = -\frac{1}{2}$$

$$\lambda_7\lambda_7 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow d_{778} = -\frac{1}{2\sqrt{3}}, \; d_{773} = -\frac{1}{2}$$

$$\lambda_8\lambda_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = a\mathbb{1} + b\lambda_8$$

$$1 = 3a + \sqrt{3}\,b$$

$$4 = 3a - 2\sqrt{3}\,b$$

$$6 = 9a, \quad a = 2/3 \checkmark \qquad b = -\frac{1}{\sqrt{3}} = d_{888}$$

$$\lambda_1 \lambda_2 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\lambda_1 \lambda_2$$

$$\{\lambda_1, \lambda_2\} = 0 \quad , \quad d_{12c} = 0$$

$$[\lambda_1, \lambda_2] = i\,2\lambda_3 \quad , \quad f_{123} = 1 = f_{231} = f_{312}$$

$$= -f_{213} = -f_{132} = -f_{321}$$

$$\lambda_1 \lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_4 \lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\{\lambda_1, \lambda_4\} = \lambda_6 \quad , \quad d_{146} = \frac{1}{2}$$

$$[\lambda_1, \lambda_4] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = i\lambda_7 \quad , \quad f_{147} = \frac{1}{2}$$

$$f_{417} = -\frac{1}{2}$$

$$\lambda_1 \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_5 \lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}$$

$$\{\lambda_1, \lambda_5\} = \lambda_7 \qquad d_{157} = \frac{1}{2}$$

$$[\lambda_1, \lambda_5] = -i\lambda_6 \quad , \quad f_{156} = -\frac{1}{2}$$

$$\lambda_1 \lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_6 \lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_1, \lambda_6\} = \lambda_4 \quad , \quad d_{164} = \frac{1}{2}$$

$$[\lambda_1, \lambda_6] = i\lambda_5 \quad , \quad f_{165} = \frac{1}{2}$$

$$\lambda_1\lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\{\lambda_1, \lambda_7\} = \lambda_5, \quad d_{175} = \frac{1}{2}$$

$$[\lambda_1, \lambda_7] = -i\lambda_4, \quad f_{174} = -\frac{1}{2}$$

$$\lambda_1\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8\lambda_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_1, \lambda_8\} = \frac{2}{\sqrt{3}}\lambda_1, \quad d_{181} = \frac{1}{\sqrt{3}}$$

$$[\lambda_1, \lambda_8] = 0$$

$$\lambda_2\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4\lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

$$\{\lambda_2, \lambda_4\} = -\lambda_7, \quad d_{247} = -\frac{1}{2}$$

$$[\lambda_2, \lambda_4] = i\lambda_6, \quad f_{246} = \frac{1}{2}$$

$$\lambda_2\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5\lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\{\lambda_2, \lambda_5\} = \lambda_6, \quad d_{256} = \frac{1}{2}$$

$$[\lambda_2, \lambda_5] = i\lambda_7, \quad f_{257} = \frac{1}{2}$$

$$\lambda_2\lambda_6 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6\lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\{\lambda_2, \lambda_6\} = \lambda_5, \quad d_{265} = \frac{1}{2}$$

$$[\lambda_2, \lambda_6] = -i\lambda_4, \quad f_{264} = -\frac{1}{2}$$

$$\lambda_2\lambda_7 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7\lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_2, \lambda_7\} = -\lambda_4, \quad d_{274} = -\frac{1}{2}$$

$$[\lambda_2, \lambda_7] = -i\lambda_5, \quad f_{275} = -\frac{1}{2}$$

$$\lambda_2\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8\lambda_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_2, \lambda_8\} = \frac{2}{\sqrt{3}}\lambda_2, \quad d_{282} = \frac{1}{\sqrt{3}}$$

$$\lambda_3\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4\lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_3, \lambda_4\} = \lambda_4, \quad d_{344} = \frac{1}{2}$$

$$[\lambda_3, \lambda_4] = i\lambda_5, \quad f_{345} = \frac{1}{2}$$

$$\lambda_3\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5\lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\{\lambda_3, \lambda_5\} = \lambda_5, \quad d_{355} = \frac{1}{2}$$

$$[\lambda_3, \lambda_5] = -i\lambda_4, \quad f_{354} = -\frac{1}{2}$$

$$\lambda_3\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6\lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\{\lambda_3, \lambda_6\} = -\lambda_6, \quad d_{366} = -\frac{1}{2}$$

$$[\lambda_3, \lambda_6] = -i\lambda_7, \quad f_{367} = -\frac{1}{2}$$

$$\lambda_3 \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

$$\{\lambda_3, \lambda_7\} = -\lambda_7, \quad d_{377} = -\frac{1}{2}$$

$$[\lambda_3, \lambda_7] = i\lambda_6, \quad f_{376} = \frac{1}{2}$$

$$\lambda_3 \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \lambda_3 = \lambda_8 \lambda_3$$

$$\{\lambda_3, \lambda_8\} = \frac{2}{\sqrt{3}} \lambda_3, \quad d_{383} = \frac{1}{\sqrt{3}}$$

$$\lambda_4 \lambda_5 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \lambda_5 \lambda_4 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$$

$$\{\lambda_4, \lambda_5\} = 0$$

$$[\lambda_4, \lambda_5] = 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{2i}{2} (\lambda_3 + \sqrt{3}\lambda_8)$$

$$f_{453} = \frac{1}{2}, \quad f_{458} = \frac{\sqrt{3}}{2}$$

$$\lambda_4 \lambda_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 \lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_4, \lambda_6\} = \lambda_1, \quad d_{461} = \frac{1}{2}$$

$$[\lambda_4, \lambda_6] = i\lambda_2, \quad f_{462} = \frac{1}{2}$$

$$\lambda_4\lambda_7 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_4, \lambda_7\} = -\lambda_2, \quad d_{472} = -\frac{1}{2}$$

$$[\lambda_4, \lambda_7] = i\lambda_1, \quad f_{471} = \frac{1}{2}$$

$$\lambda_4\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_8\lambda_4 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_4, \lambda_8\} = -\frac{1}{\sqrt{3}}\lambda_4, \quad d_{484} = -\frac{1}{2\sqrt{3}}$$

$$[\lambda_4, \lambda_8] = -\sqrt{3}\, i\lambda_5, \quad f_{485} = -\frac{\sqrt{3}}{2}$$

$$\lambda_5\lambda_6 = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_5, \lambda_6\} = \lambda_2, \quad d_{562} = \frac{1}{2}$$

$$[\lambda_5, \lambda_6] = -i\lambda_1, \quad f_{561} = -\frac{1}{2}$$

$$\lambda_5\lambda_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\{\lambda_5, \lambda_7\} = \lambda_1, \quad d_{571} = \frac{1}{2}$$

$$[\lambda_5, \lambda_7] = i\lambda_2, \quad f_{572} = \frac{1}{2}$$

$$\lambda_5\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_8\lambda_5 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix}$$

$$\{\lambda_5, \lambda_8\} = -\frac{1}{\sqrt{3}}\lambda_5, \quad d_{585} = -\frac{1}{2\sqrt{3}}$$

$$[\lambda_5, \lambda_8] = \sqrt{3}\, i\lambda_4, \quad f_{584} = \frac{\sqrt{3}}{2}$$

$$\lambda_6\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \lambda_7\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$$

$$\{\lambda_6, \lambda_7\} = 0, \quad [\lambda_6, \lambda_7] = i(\sqrt{3}\lambda_8 - \lambda_3)$$

$$f_{678} = \frac{\sqrt{3}}{2}, \quad f_{673} = -\frac{1}{2}$$

$$\lambda_6\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_8\lambda_6 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\{\lambda_6, \lambda_8\} = -\frac{1}{\sqrt{3}}\lambda_6, \quad d_{686} = -\frac{1}{2\sqrt{3}}$$

$$[\lambda_6, \lambda_8] = -i\sqrt{3}\,\lambda_7, \quad f_{687} = -\frac{\sqrt{3}}{2}$$

$$\lambda_7\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8\lambda_7 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -2i & 0 \end{pmatrix}$$

$$\{\lambda_7, \lambda_8\} = -\frac{1}{\sqrt{3}}\lambda_7, \quad d_{787} = -\frac{1}{2\sqrt{3}}$$

$$[\lambda_7, \lambda_8] = \sqrt{3}\, i\lambda_6, \quad f_{786} = \frac{\sqrt{3}}{2}$$

We see that f is totally antisymmetric while d is totally symmetric. The independent, non-zero ones are:

$$\frac{1}{2} = \frac{1}{2} f_{123} = f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367},$$

$$\frac{\sqrt{3}}{2} = f_{458} = f_{678}$$

$$\frac{1}{\sqrt{3}} = d_{118} = d_{228} = d_{338} = -d_{888}$$

$$-\frac{1}{2\sqrt{3}} = d_{448} = d_{558} = d_{668} = d_{778}$$

$$\frac{1}{2} = d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377}$$