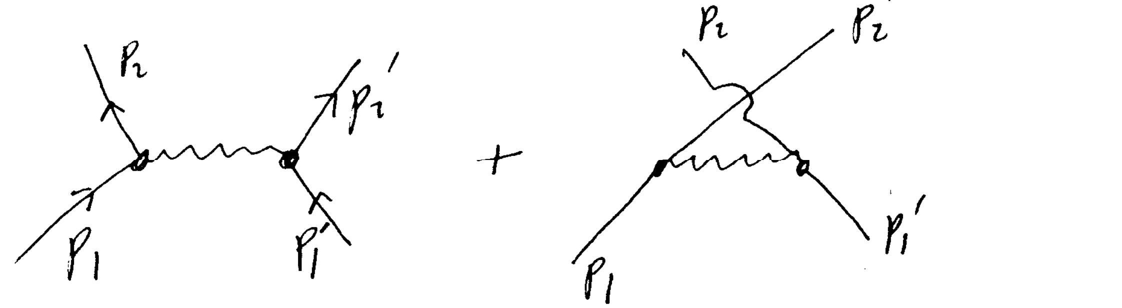
Homework #7

7-1



$$\Big\{ \bar{u}_{p_1}(-ie)\gamma^\mu u_{p_2} \frac{-i}{(p_2-p_1)^2} \bar{u}_{p_1'}(-ie)\gamma_\mu u_{p_2'}$$

"kinematic factor"

cross ext lines

$$\pm \bar{u}_{p_1}(-ie)\gamma^\mu u_{p_2'} \frac{-i}{(p_2'-p_1)^2} \bar{u}_{p_1'}(-ie)\gamma_\mu u_{p_2} \Big\} (2m)^2\sqrt{d\tilde{p}_1\, d\tilde{p}_1'\, d\tilde{p}_2 \ldots}$$

$$= ie^2 \Big\{ \frac{(\bar{u}_{p_1}\gamma^\mu u_{p_2})(\bar{u}_{p_1'}\gamma_\mu u_{p_2'})}{(p_1-p_2)^2} - \frac{(\bar{u}_{p_1}\gamma^\mu u_{p_2'})(\bar{u}_{p_1'}\gamma_\mu u \ldots}{(p_1-p_2')^2}$$

$$\times (2m)^2\sqrt{d\tilde{p}_1\, d\tilde{p}_1'\, d\tilde{p}_2\, d\tilde{p}_2'}$$

7-2 Now, the boost eqn. from p. 28 of the notes is

$$u_{p\sigma} = \frac{1}{\sqrt{2m}}\left[\sqrt{E+m} + \sqrt{E-m}\, i\gamma_5\sigma\right] v_\sigma$$

$$\simeq \sqrt{\frac{E}{2m}}\,(1 + i\gamma_5\sigma)\, v_\sigma \text{ if } E = \frac{M}{2} \gg m$$

Then $u^*_{p_1\sigma_1} u_{p_2\sigma_2} = \frac{M/2}{2m} v^*_{\sigma_1}(1 + i\gamma_5(\sigma_1+\sigma_2) + \sigma_1\sigma_2) v_{\sigma_2}$

But $v^*_{\sigma_1} i\gamma_5 v_{\sigma_2} = v^*_{\sigma_1} i\gamma_5\gamma^0 v_{\sigma_2} = -v^*_{\sigma_1}\gamma^0 i\gamma_5 v_{\sigma_2}$

$$= -v^*_{\sigma_1} i\gamma_5 v_{\sigma_2} = 0$$

so $u^*_{p_1\sigma_1} u_{p_2\sigma_2} = \begin{cases} \frac{M}{2m} v^*_{\sigma_1} v_{\sigma_2} & \sigma_1 = \sigma_2 \\ 0 & \sigma_1 = -\sigma_2 \end{cases}$
(to order (M/m))

On the other hand

$$u^*_{p_1\sigma_1} \underbrace{\gamma^0 \vec{\gamma}}_{i\gamma_5\vec{\sigma}} u_{p_2\sigma_2} = \frac{M/2}{2m} v^*_{\sigma_1}(1 + i\gamma_5\sigma_1)\, i\gamma_5\vec{\sigma} \times (1 + i\gamma_5\sigma_2) v_{\sigma_2}$$

$$= \frac{M}{2m}\,\frac{1}{2}(\sigma_1 + \sigma_2)\, v^*_{\sigma_1}\vec{\sigma}\, v_{\sigma_2}$$

which also vanishes if $\sigma_1 = -\sigma_2$.

Thus, if $\sigma_1 = \sigma_2$; $\sigma_1' = \sigma_2'$, the first numerator in 7-1 is
$(\bar{u}_{p_1\sigma_1}\gamma^\mu u_{p_2\sigma_2})(\bar{u}_{p_1'\sigma_1'}\gamma_\mu u_{p_2'\sigma_2'})$

$$(*) = \left(\frac{M}{2m}\right)^2 \left[-(v^*_{\sigma_1} v_{\sigma_2})(v^*_{\sigma_1'} v_{\sigma_2'}) + \sigma_1\sigma_1' (v^*_{\sigma_1}\vec{\sigma} v_{\sigma_2})\cdot(v^*_{\sigma_1'}\vec{\sigma} v_{\sigma_2'}) \right]$$

Now any 2×2 matrix can be written as a linear combination of $1, \sigma_x, \sigma_y, \sigma_z$, or

$$X = a_\mu \sigma^\mu \qquad \sigma^\mu = (1, \vec{\sigma})$$

$$\mathrm{Tr}\,\sigma^\lambda X = a_\mu \mathrm{Tr}\,\sigma^\lambda\sigma^\mu = a_\mu 2\delta^{\lambda\mu} = 2a_\lambda$$

$$X = \frac{1}{2}\sigma^\mu \mathrm{Tr}(\sigma^\mu X)$$

Thus $\mathrm{Tr}(\mathcal{X}\,\mathcal{Y}) = \frac{1}{4}\,\mathrm{Tr}\,\sigma^\mu\sigma^\nu\;\mathrm{Tr}(\sigma^\mu\mathcal{X})\,\mathrm{Tr}(\sigma^\nu\mathcal{Y})$

$$= \frac{1}{2}\,\delta^{\mu\nu}\,\mathrm{Tr}\,\sigma^\mu\mathcal{X}\;\mathrm{Tr}\,\sigma^\nu\mathcal{Y}$$

$$= \frac{1}{2}\left[\mathrm{Tr}\,\mathcal{X}\;\mathrm{Tr}\,Y + \mathrm{Tr}(\vec{\sigma}\mathcal{X})\cdot\mathrm{Tr}(\vec{\sigma}\mathcal{Y})\right]$$

$$= \frac{1}{2}\left[\mathrm{Tr}(\vec{\sigma}\mathcal{X})\cdot\mathrm{Tr}(\vec{\sigma}Y) - \mathrm{Tr}\,\mathcal{X}\;\mathrm{Tr}\,Y\right]$$

$$+ \;\mathrm{Tr}\,\mathcal{X}\;\mathrm{Tr}\,Y$$

or $\boxed{(\mathrm{Tr}\,\sigma^\mu\mathcal{X})(\mathrm{Tr}\,\sigma_\mu Y) = 2\left[\mathrm{Tr}\,\mathcal{X}Y - \mathrm{Tr}\,\mathcal{X}\,\mathrm{Tr}\,Y\right]}$

Thus, for <u>equal incident helicities</u>, $\sigma_1 = \sigma_1'$ (and hence $\sigma_2 = \sigma_2'$) the square bracket in (*) is

$$(v_{\sigma_1}^*\vec{\sigma}\,v_{\sigma_2})\cdot(v_{\sigma_1'}^*\,\vec{\sigma}\,v_{\sigma_2'}) - (v_{\sigma_1}^*\,v_{\sigma_2})(v_{\sigma_1'}^*\,v_{\sigma_2'})$$

$$= \mathrm{Tr}(\vec{\sigma}\,\underbrace{v_{\sigma_2}v_{\sigma_1}^*}_{\mathcal{X}})\cdot\mathrm{Tr}(\vec{\sigma}\,\underbrace{v_{\sigma_2'}v_{\sigma_1'}^*}_{Y}) - \mathrm{Tr}(v_{\sigma_2}v_{\sigma_1}^*)\,\mathrm{Tr}(v_{\sigma_2'}v_{\sigma_1'}^*)$$

$$= 2\left[\mathrm{Tr}(v_{\sigma_2}v_{\sigma_1}^*\,v_{\sigma_2'}v_{\sigma_1'}^*) - \mathrm{Tr}(v_{\sigma_2}v_{\sigma_1}^*)\,\mathrm{Tr}(v_{\sigma_2'}v_{\sigma_1'}^*)\right]$$

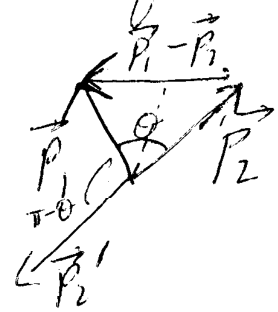
$$= 2\left[(v_{\sigma_1'}^*v_{\sigma_2})(v_{\sigma_1}^*v_{\sigma_2'}) - (v_{\sigma_1}^*v_{\sigma_2})(v_{\sigma_1'}^*v_{\sigma_2'})\right]$$

Now the second numerator in amplitude in 7-1 differs from the first by $\sigma_2 \leftrightarrow \sigma_2'$; which just changes the sign of the above result. So we have the amplitude

$$ie^2 2\left[(v_{\sigma_1'}^* v_{\sigma_2})(v_{\sigma_1}^* v_{\sigma_2'}) - (v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1'}^* v_{\sigma_2'})\right]$$

$$\times \left[\frac{1}{(P_1 - P_2)^2} + \frac{1}{(P_1 - P_2')^2}\right]\left(\frac{M}{2m}\right)^2$$

Now the last factor involves in the CM frame



$$(P_1 - P_2)^2 = (\vec{P}_1 - \vec{P}_2)^2$$

$$= \left[2|\vec{P}_1| \sin\frac{\theta}{2}\right]^2$$

$$= M^2 \sin^2\frac{\theta}{2}$$

$$(P_1 - P_2')^2 = (\vec{P}_1 - \vec{P}_2')^2 = \left[2|P_1| \sin\frac{\pi-\theta}{2}\right]^2 = M^2 \cos^2\frac{\theta}{2}$$

So $$\frac{1}{(P_1 - P_2)^2} + \frac{1}{(P_1 - P_2')^2} = \frac{1}{M^2}\left[\frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{\sin^2\frac{\theta}{2}}\right]$$

To evaluate the spinors, choose $\vec{P}_1 = -\vec{P}_1'$ to be the reference direction. Then we have two possibilities. Suppose

$$v_{\sigma_1} = v_+ = \begin{pmatrix}1\\0\end{pmatrix} \text{ so } v_{\sigma_1'} = v_- = \begin{pmatrix}0\\1\end{pmatrix}$$

$$(\vec{\sigma}\cdot\hat{P}_1\, v_{\sigma_1} = v_{\sigma_1},\quad \vec{\sigma}\cdot\hat{P}_1' v_{\sigma_1'} = -\vec{\sigma}\cdot\hat{P}_1 v_{\sigma_1'} = v_{\sigma_1'})$$

The the outgoing spinors refer to the direction of $\vec{P}_2 = -\vec{P}_2'$ which is rotated by an angle θ about the x axis (say):

$$v_{\sigma_2} = e^{i\theta\frac{1}{2}\sigma_x} v_+ = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}$$

$$v_{\sigma_2'} = e^{i\theta\frac{1}{2}\sigma_x} v_- = \begin{pmatrix} i\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

so the spinor factor in the amplitude on the top of p. 4 is

$$(v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1}^* v_{\sigma_2'}) - (v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1'}^* v_{\sigma_2'})$$

$$= -(\cos\theta/2)(\cos\theta/2) + (i\sin\theta/2)(i\sin\theta/2)$$

$$= -\cos^2\theta/2 - \sin^2\theta/2 = -1$$

and our result for the amplitude is (apart from kinematic factor)

$$ie^2\, 2(-1)\frac{M^2}{4m^2}\frac{1}{M^2}\left[\frac{1}{\cos^2\theta/2} + \frac{1}{\sin^2\theta/2}\right]$$

$$= \boxed{-\frac{ie^2}{2m^2}\left[\frac{1}{\cos^2\theta/2} + \frac{1}{\sin^2\theta/2}\right]} \qquad \begin{matrix} \sigma_1 = \sigma_1' \\ M \gg m \end{matrix}$$

[Evidently, the same result follows if we had chosen $v_{\sigma_1} = v_-$, for the effect would be to interchange the labels $\sigma_1' \leftrightarrow \sigma_1$, $\sigma_2' \leftrightarrow \sigma_2$ which leaves the amplitude on the top of p. 4 unchanged.]

7-3 In general, the incident flux is given by

$$F = 2\, d\tilde{p}_a\, d\tilde{p}_b\, [M^2-(m_a+m_b)^2]^{1/2}[M^2-(m_a-m_b)^2]^{1/2}$$

for two incident particles of momentum $\vec{p}_a$ & $\vec{p}_b$ and masses m_a & m_b. In the high energy limit $M >> m_a, m_b$ so

$$F \to 2\, d\tilde{p}_a\, d\tilde{p}_b\, M^2$$

As for the final state, we must integrate over the momentum-conserving δ function

$$I = \int d\tilde{p}_a\, d\tilde{p}_b\, (2\pi)^4\, \delta(p_a+p_b-P) \qquad P^2=-M^2$$

$$= \frac{1}{4(2\pi)^2}\int \frac{(d\vec{p}_a)}{p_a^0}\frac{(d\vec{p}_b)}{p_b^0}\, \delta(\vec{p}_a+\vec{p}_b)\, \delta(p_a^0+p_b^0-M)$$

$$p^0 = \sqrt{\vec{p}^{\,2}+m^2}$$

Since $\vec{p} = \vec{p}_a = -\vec{p}_b$, $\quad \sqrt{\vec{p}^{\,2}+m_a^2} + \sqrt{\vec{p}^{\,2}+m_b^2} = M$

or $\cancel{\vec{p}^{\,2}}+m_a^2 = \left(M-\sqrt{\vec{p}^{\,2}+m_b^2}\right)^2 = M^2 - 2M\sqrt{\vec{p}^{\,2}+m_b^2} + \cancel{\vec{p}^{\,2}}+m_b^2$

$$2M(\vec{p}^{\,2}+m_b^2)^{1/2} = M^2+m_b^2-m_a^2$$

$$4M^2(\vec{p}^{\,2}+m_b^2) = (M^2+m_b^2-m_a^2)^2$$

$$\vec{p}^{\,2} = \frac{(M^2+m_b^2-m_a^2)^2}{4M^2} - m_b^2 = \frac{(M^4-2M^2(m_a^2+m_b^2) + (m_b^2-m_a^2)^2)}{4M^2}$$

$$= \frac{1}{4M^2}[M^2-(m_b+m_a)^2][M^2-(m_b-m_a)^2]$$

or $|\vec{p}| = \frac{1}{2M}\left[M^2 - (m_a + m_b)^2\right]^{1/2}\left[M^2 - (m_a - m_b)^2\right]^{1/2}$.

So $I = \frac{1}{16\pi^2}\int \frac{(d\vec{p}_a)}{p_a^0\, p_b^0}\,\delta(p_a^0 + p_b^0 - M)$

where $(d\vec{p}_a) = |\vec{p}|^2 d|\vec{p}|\, d\Omega$

$$d|\vec{p}| = d(p_a^0 + p_b^0)\,\frac{1}{\frac{\partial}{\partial|\vec{p}|}(p_a^0 + p_b^0)} = \frac{d(p_a^0 + p_b^0)}{\frac{|\vec{p}|}{p_a^0} + \frac{|\vec{p}|}{p_b^0}}$$

$$\left(\frac{dp^0}{d|\vec{p}|} = \frac{|\vec{p}|}{p_0}\right) \qquad = \frac{1}{|\vec{p}|}\,\frac{p_a^0 p_b^0}{p_a^0 + p_b^0}\, d(p_a^0 + p_b^0)$$

$$\therefore I = \frac{1}{16\pi^2}\,|\vec{p}|\,\frac{1}{M}\,d\Omega\int d(p_a^0 + p_b^0)\,\delta(p_a^0 + p_b^0 - M)$$

$$= \frac{1}{32\pi^2}\,\frac{d\Omega}{M^2}\left[M^2 - (m_a + m_b)^2\right]^{1/2}\left[M^2 - (m_a - m_b)^2\right]^{1/2}$$

Therefore, for elastic scattering, where the initial & final particles are the same,

$$\frac{I}{F} = \frac{1}{(2\, d\tilde{p}_1\, d\tilde{p}_1')}\,\frac{d\Omega}{32\pi^2}\,\frac{1}{M^2}$$

So, from the amplitude found in 7-2, we find

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{1}{2}}_{\text{kinematic factor}}\frac{e^4}{4m^4}\,\frac{1}{32\pi^2}\,\frac{1}{M^2}\left(\frac{1}{\cos^2\theta/2} + \frac{1}{\sin^2\theta/2}\right)^2 16m^4$$

or with $\alpha = e^2/4\pi$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{M^2}\left(\frac{1}{\sin^2\theta/2} + \frac{1}{\cos^2\theta/2}\right)^2} \quad \sigma_1 = \sigma_1', \; M \gg m$$

7-4 What if $\sigma_1' = -\sigma_1$? Then in the high energy limit $\sigma_2' = -\sigma_2$, and $(*)$ on p.2 becomes

$$\left(\frac{M}{2m}\right)^2\left[-(v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1'}^* v_{\sigma_2'}) - (v_{\sigma_1}^* \vec{\sigma}\, v_{\sigma_2})(v_{\sigma_1'}^* \vec{\sigma}\, v_{\sigma_2'})\right]$$

$$= \left(\frac{M}{2m}\right)^2\Big[-\cancel{(v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1'}^* v_{\sigma_2'})}$$

From p.3

$$-\left(2(v_{\sigma_1'}^* v_{\sigma_2})(v_{\sigma_1}^* v_{\sigma_2'}) - \cancel{(v_{\sigma_1}^* v_{\sigma_2})(v_{\sigma_1'}^* v_{\sigma_2'})}\right)\Big]$$

$$= -2\left(\frac{M}{2m}\right)^2 (v_{\sigma_1'}^* v_{\sigma_2})(v_{\sigma_1}^* v_{\sigma_2'}).$$

So the amplitude, apart from the kinematic factor is

$$-2ie^2\frac{M^2}{4m^2}\left\{\frac{(v_{\sigma_1'}^* v_{\sigma_2})(v_{\sigma_1}^* v_{\sigma_2'})}{(p_1-p_2)^2} - \frac{(v_{\sigma_1'}^* v_{\sigma_2'})(v_{\sigma_1}^* v_{\sigma_2})}{(p_1-p_2')^2}\right\}$$

where if $v_{\sigma_1} = \begin{pmatrix}1\\0\end{pmatrix} = v_+ \qquad v_{\sigma_1'} = v_+ = \begin{pmatrix}1\\0\end{pmatrix}$

For the 1st term $v_{\sigma_2} = e^{i\theta\frac{1}{2}\sigma_x} v_+ = \begin{pmatrix}\cos\theta/2\\ i\sin\theta/2\end{pmatrix} = v_{\sigma_2'}$
$\sigma_1 = \sigma_2 = -\sigma_1' = -\sigma_2'$ (see p.2)

For the 2nd term $v_{\sigma_2'} = e^{i\theta\frac{1}{2}\sigma_x} v_- = \begin{pmatrix} i\sin\theta/2\\ \cos\theta/2\end{pmatrix} = v_{\sigma_2}$
$\sigma_1 = \sigma_2' = -\sigma_1' = -\sigma_2$

so $(v_{\sigma_1}^{*}v_{\sigma_2})(v_{\sigma_1}^{*}v_{\sigma_2'}) = \cos^2\theta/2$

$(v_{\sigma_1}^{*}v_{\sigma_2'})(v_{\sigma_1}^{*}v_{\sigma_2}) = -\sin^2\theta/2$

The two terms <u>do not interfere</u> since they are distinct final states, so we can immediately write down the cross section by squaring + adding the amplitudes:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{M^2}\left[\frac{\cos^4\theta/2}{\sin^4\theta/2} + \frac{\sin^4\theta/2}{\cos^4\theta/2}\right]}$$

$\sigma_1 = -\sigma_1'$

$M \gg m$

The <u>unpolarized</u> cross section is obtained <u>averaging</u> this result with that of problem 7-3

The result is $(s = \sin\theta/2,\ c = \cos\theta/2)$

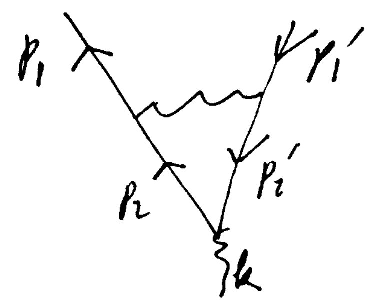
$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{\alpha^2}{M^2}\left\{\frac{1}{2}\left(\frac{1}{s^2}+\frac{1}{c^2}\right)^2 + \frac{1}{2}\left(\frac{\overbrace{c^4}^{(1-s^2)^2}}{s^4} + \frac{\overbrace{s^4}^{(1-c^2)^2}}{c^4}\right)\right\}$$

$$= \frac{\alpha^2}{M^2}\left\{\frac{1}{2}\left(\frac{1}{s^4}+\frac{1}{c^4}+\frac{2}{s^2c^2}+\frac{1}{s^4}-\frac{2}{s^2}+1 + \frac{1}{c^4}-\frac{2}{c^2}+1\right)\right\}$$

$$= \frac{\alpha^2}{M^2}\left\{\frac{1}{s^4}+\frac{1}{c^4}+\left(\frac{1}{s^2}+\frac{1}{c^2}\right)-\frac{1}{s^2}-\frac{1}{c^2}+1\right\}$$

$$= \frac{\alpha^2}{M^2}\left\{\frac{1}{s^4}+\frac{1}{c^4}+1\right\} = \frac{\alpha^2}{M^2}\left[\frac{1}{\sin^2\theta/2}+\frac{1}{\cos^2\theta/2}-1\right]^2$$

7.5



$$\text{Amplitude} = \bar{\psi}(-p_1)(-ie)\gamma^{\mu}\frac{-i}{m+\gamma p_2}(-ie)\gamma^{\lambda}A_{\lambda}(k)\frac{-i}{m-\gamma p_2'}$$

$$\times(-ie)\gamma_{\mu}\,\psi(-p_1')\frac{-i}{(p_1-p_2)^2}$$

$$= -e^3\bar{\psi}(-p_1)\gamma^{\mu}\frac{m-\gamma p_2}{m^2+p_2^2}\gamma^{\lambda}A_{\lambda}(k)\frac{m+\gamma p_2'}{m^2+p_2'^2}\gamma_{\mu}\psi(-p_1')$$

$$\times\frac{1}{(p_1-p_2)^2}$$

Now the Dirac matrix structure is

$$\gamma^{\mu}(m-\gamma p_2)\gamma^{\lambda}(m+\gamma p_2')\gamma_{\mu}$$

We encounter

$$\gamma^{\mu}\gamma^{\lambda}\gamma_{\mu} = -2\gamma^{\mu} - \underbrace{\gamma^{\mu}\gamma_{\mu}}_{-4}\gamma^{\lambda} = 2\gamma^{\lambda}$$

which we drop since this is not a magnetic moment term. Next we see

$$m\gamma^{\mu}(\gamma^{\lambda}\gamma p_2' - \gamma p_2\gamma^{\lambda})\gamma_{\mu} = m\left[-\gamma^{\lambda}\gamma^{\mu}\gamma p_2'\gamma_{\mu}\right.$$

$$\left.-2g^{\lambda\mu}\gamma p_2'\gamma_{\mu} + \gamma^{\mu}\gamma p_2\gamma_{\mu}\gamma^{\lambda} + 2g^{\lambda}_{\ \mu}\gamma^{\mu}\gamma p_2\right]$$

$$= m\left[-2\gamma^{\lambda}\gamma p_2' - 2\gamma p_2'\gamma^{\lambda} + 2\gamma p_2\gamma^{\lambda} + 2\gamma^{\lambda}\gamma p_2\right]$$

$$= 4m\left[p_2'^{\lambda} - p_2^{\lambda}\right]$$

Finally we have

$$-\gamma^\mu \gamma p_2 \gamma^\lambda \gamma p_2' \gamma_\mu$$

$$= [+2p_2^\mu + \gamma p_2 \gamma^\mu]\, \gamma^\lambda\, [-2p'_{2\mu} - \gamma_\mu \gamma p_2']$$

$$= -4 p_2 p_2' \gamma^\lambda - 2\gamma^\lambda \gamma p_2 \gamma p_2' - 2\gamma p_2 \gamma p_2' \gamma^\lambda$$

$$(*) \qquad - \gamma p_2\, 2\gamma^\lambda \gamma p_2'$$

Now we make use of the external projection factors: on the left $m + \gamma p_1 = 0$, and on the right $m - \gamma p_1' = 0$. Then

$$-2\gamma^\lambda \gamma p_2 \gamma p_2' = -2\gamma^\lambda [\gamma p_2 - \gamma p_1 + \gamma p_1]$$

$$\text{(because } p_1 - p_2 = p_2' - p_1'\text{)} \qquad \times [\gamma(p_1 - p_2) + m]$$

$$= -2\gamma^\lambda [(p_1 - p_2)^2 + \gamma p_1 \gamma(p_1 - p_2) + m\gamma p_2]$$

where
$$-2\gamma^\lambda \gamma p_1 \gamma(p_1 - p_2) = +4p_1^\lambda \gamma(p_1 - p_2) - 2m\gamma^\lambda \gamma(p_1 - p_2)$$

and
$$-2\gamma^\lambda m \gamma p_2 = 4mp_2^\lambda + 2m(\gamma(p_2 - p_1) - m)\gamma^\lambda$$

As noted above terms with only the single Dirac matrix γ^λ are electric monopole terms (like the initial interaction) rather than magnetic dipole, so we drop these. We are left with, from the 2nd term in (*):

$$-2\gamma^\lambda \gamma p_2 \gamma p_2' \to 4p_1^\lambda \gamma(p_1-p_2) + 4mp_2^\lambda$$
$$-2m\underbrace{\{\gamma^\lambda, \gamma(p_1-p_2)\}}_{-2(p_1-p_2)^\lambda}$$
$$= 4p_1^\lambda \gamma(p_1-p_2) + 4mp_1^\lambda$$

The 3rd term in (*) is reduced similarly:

$$-2\gamma p_2 \gamma p_2' \gamma^\lambda = -2[\gamma(p_1'-p_2') - m][\gamma(p_2'-p_1') + \gamma p_1' \ldots$$
$$= -2[(p_1'-p_2')^2 + \gamma(p_1'-p_2')\gamma p_1' - m\gamma p_2']\gamma \ldots$$
$$\to -2[-2p_1'^\lambda \gamma(p_1'-p_2') - m\gamma(p_1'-p_2')\gamma^\lambda$$
$$+ 2mp_2'^\lambda + m\gamma^\lambda[\gamma(p_2'-p_1') + m]$$
$$\to 4p_1'^\lambda \gamma(p_1'-p_2') - 4mp_2'^\lambda - 4m(p_1'-p_2')$$
$$= 4p_1'^\lambda \gamma(p_1'-p_2') - 4mp_1'^\lambda$$

The 4th term in (*) is

$$-2\gamma p_2 \gamma^\lambda \gamma p_1' = -2[\gamma(p_2-p_1) - m]\gamma^\lambda[\gamma(p_1'-p_1 \ldots$$
$$\to -2[-2(p_2-p_1)^\lambda \gamma(p_1-p_2) + 2m(p_1-p_2)^\lambda]$$

So altogether we find for the Dirac structure

$$\gamma^\mu(m-\gamma p_1)\gamma^\lambda(m+\gamma p_2')\gamma_\mu \to 4m(p_1'-p_2)^\lambda \; (p \ldots$$
$$+ \cancel{4p_1^\lambda \gamma(p_1-p_2)} + \cancel{4mp_1^\lambda}$$
$$+ 4p_1'^\lambda \gamma(p_2-p_1) - 4mp_1'^\lambda + 4(p_2-p_1)^\lambda \gamma(p_1-p_2) \; \underline{-4m(p_1- \ldots}$$
$$= 4m(p_1-p_2)^\lambda + 4(p_2-p_1')^\lambda \gamma(p_1-p_2)$$

So we have to evaluate the integ.:

$$I^\mu = \int \frac{(dp_2)}{(2\pi)^4} \frac{(p_1-p_2)^\mu}{(m^2+p_2^2)(m^2+p_2'^2)(p_1-p_2)^2}$$

and $$I^{\mu\nu} = \int \frac{(dp_2)}{(2\pi)^4} \frac{(p_1-p_2)^\mu (p_1-p_2)^\nu}{(m^2+p_2^2)(m^2+p_2'^2)(p_1-p_2)^2}$$

where $p_2' = k - p_2$, $p_1^2 = p_1'^2 = -m^2$, $k^2 = 0$

Now $I^\mu \to -I^\mu$ under $p_1 \leftrightarrow p_1'$ $\left(p_1 - p_2 = p_2' - p_1' \right.$

$\left. \quad + \int \frac{(dp_2)}{(2\pi)^4} = \int \frac{(dp_2)(dp_2')}{(2\pi)^8} (2\pi)^4 \delta(p_2 + p_2' - k) \right)$

$\therefore \quad I^\mu = a\,(p_1 - p_1')^\mu$ where a is a scalar.

$$p_1^\mu I_\mu = a\,(p_1 - p_1')p_1 = a\,(p_1^2 - p_1 p_1')$$
$$= a\left(p_1^2 + p_1'^2 - \tfrac{1}{2}\underbrace{(p_1+p_1')^2}_{k^2}\right) = a(-2m^2)$$

But

$$p_1^\mu (p_1 - p_2)_\mu = p_1^2 - p_1 p_2 = \tfrac{1}{2}p_1^2 - \frac{p_2^2}{2} + \tfrac{1}{2}(p_1-p_2)^2 = -\tfrac{1}{2}(p_2^2+m^2) + \tfrac{1}{2}(p_1-p_2)^2$$

$$\therefore \quad a = -\frac{1}{2m^2}\,\frac{1}{2}\left\{ \int \frac{(dp_2)}{(2\pi)^4} \frac{1}{(p_2^2+m^2)(p_2'^2+m^2)} - \int \frac{(dp_2)}{(2\pi)^4} \frac{1}{(p_2'^2+m^2)(p_1 - \right.$$

Similarly

$$I^{\mu\nu} = A g^{\mu\nu} + B k^\mu k^\nu + C(p_1 - p_1')^\mu (p_1 - p_1')^\nu$$

$$k_\mu I^{\mu\nu} = A k^\nu + B k^2 k^\nu = (A + Bk^2) k^\nu$$

while $k_\mu (p_1 - p_2)^\mu = (p_1 + p_1') p_1 - (p_2 + p_1') p_2$

$$= \tfrac{1}{2}(p_1 + p_1')^2 - \tfrac{1}{2}(p_2 + p_1')^2 + \tfrac{1}{2}(p_2'^2 - \ldots$$

$$= \tfrac{1}{2}(k^2) - \tfrac{1}{2} k^2 + \tfrac{1}{2}(p_2'^2 - p_2^2)$$

$$\text{so } k^\nu (A + Bk^2) = \frac{1}{2} \int \frac{(dp_2)}{(2\pi)^4} \frac{1}{(p_1 - p_2)^2} \left\{ \frac{1}{m^2 + p_2^2} - \frac{1}{m^2 + p_2'^2} \right\} (p_1 - p_2)$$

$$\text{Trace: } 4A + Bk^2 + C(p_1 - p_1')^2 = 3A + C[2p_1^2 + 2p_1'^2 - k \ldots] + (A + Bk^2)$$

$$= 3A + (-4m^2)C + (A + Bk^2)$$

$$= \int \frac{(dp_2)}{(2\pi)^4} \frac{1}{(p_2^2 + m^2)(p_2'^2 + m^2)}$$

Multiplication with $(p_1 - p_1')^\mu$:

$$[A + C(p_1 - p_1')^2](p_1 - p_1')^\nu$$

$$= [A - 4m^2 C](p_1 - p_1')^\nu$$

while $(p_1 - p_1')(p_1 - p_2) = p_1(p_1 - p_2) + p_1'(p_1' - p_2')$

$$= \tfrac{1}{2}(p_1 - p_2)^2 - \tfrac{1}{2}(p_2^2 + m^2) + \tfrac{1}{2}(p_1' - p_2')^2 - \tfrac{1}{2}(p_2'^2 + m^2)$$

$$(A - 4m^2 C)(p_1 - p_1')^{\mu}$$

$$= \frac{1}{2}\int \frac{(dp_2)}{(2\pi)^4}\left\{ \frac{2}{(p_2^2+m^2)(p_2'^2+m^2)} - \frac{1}{p_2'^2+m^2}\frac{1}{(p_1-p_2)^2} - \frac{1}{p_2^2+m^2}\frac{1}{(p_1-p_2)^2}\right\}(4$$

Now evaluate the integrals:

Combine the denominators in the usual way

$$\frac{1}{p_2^2+m^2}\,\frac{1}{(p_2-k)^2+m} = -\int_0^\infty ds\, s\int_0^1 du\; e^{-is\chi(u)}$$

$$\chi(u) = u[(p_2-k)^2+m^2] + (1-u)(p_2^2+m^2)$$

$$= p_2^2(1-u+u) - 2p_2 k\, u + m^2$$

$$= (p_2 - ku)^2 + m^2 \qquad k^2 = 0$$

so 1st integral in a is

$$-\int \frac{(dp_2)}{(2\pi)^4}\int_0^\infty ds\, s\int_0^1 du\; e^{-is[(p_2-ku)^2+m^2]}$$

$$= -\int_0^\infty ds\, s\int_0^1 du\left(-\frac{i}{16\pi^2 s^2}\right)e^{-ism^2} = +\frac{i}{16\pi^2}\int_0^\infty \frac{ds}{s}\,e^{-ism^2}$$

divergesatozc (int. log. div.)

For the 2nd integral in a

$$\chi(u) = u(p_2'-p_1')^2 + (1-u)(p_2'^2+m^2)$$
$$= p_2'^2(1-u+u) - 2u\,p_1'p_2' + m^2(1-u-u)$$
$$= (p_2' - p_1'u)^2 - p_1'^2u^2 + m^2(1-2u)$$
$$= (p_2'-p_1'u)^2 + m^2(1-u)^2$$

so the corresponding integral is

$$-\int \frac{(dp_2')}{(2\pi)^4}\int_0^\infty ds\int_0^1 du\, e^{-is[(p_2'-p_1'u)^2+m^2(1-u)^2]}$$

$$= \frac{i}{16\pi^2}\int_0^\infty \frac{ds}{s}\int_0^1 du\, e^{-ism^2(1-u)^2}$$

$$a = -\frac{1}{4m^2}\frac{i}{16\pi^2}\left\{\int_{s_0}^\infty \frac{ds}{s}e^{-ism^2} - \int_{s_0}^\infty \frac{ds}{s}\int_0^1 e^{-ism^2(1-u)^2}du\right\} s_0 \to$$
$$= +\frac{i}{64\pi^2m^2}\left(+\ln(m^2s_0) - \int_0^1 du\,\ln(m^2(1-u)^2s_0)\right)$$

$$a = \frac{-i}{64\pi^2m^2}2\int_0^1 du\,\ln(1-u) = +\frac{i}{32\pi^2m^2}$$

Similarly (p. 14)

$$+(A\cdot Bk^2) + 3A - 4m^2C = \frac{-i}{16\pi^2}\ln(m^2s_0)$$

Finally consider the integral on the top eq. .15:

$$\int \frac{(dp_2)}{(2\pi)^4} \frac{1}{p_2^2+m^2} \frac{1}{p_2'^2+m^2} (p_1-p_2)^\nu$$

$$= -\int_0^\infty ds\,s \int_0^1 du \int \frac{(dp_2)}{(2\pi)^4} (p_1-\underbrace{p_2}_{\cancel{p_2-ku}+ku})^\nu e^{-is[(p_2-ku)^2+m^2]}$$

$$= \frac{i}{16\pi^2}\int_0^\infty \frac{ds\,s}{s^2} \underbrace{\int_0^1 du\,(p_1-ku)^\nu}_{(p_1-\frac{1}{2}k)^\nu = [p_1-\frac{1}{2}(p_1+p_1')]^\nu = \frac{1}{2}(p_1-p_1')^\nu} e^{-ism^2}$$

$$= -\frac{i}{16\pi^2}\frac{1}{2}(p_1-p_1')^\nu \ln(m^2 s_0)$$

$$\int \frac{(dp_2)}{(2\pi)^4} \frac{1}{p_2^2+m^2} \frac{1}{(p_2-p_1)^2} (p_1-p_2)^\nu$$

$$= -\int_0^\infty ds\,s \int_0^1 du \int \frac{(dp_2)}{(2\pi)^4} e^{-is[(p_2-p_1u)^2+m^2(1-u)^2]} (p_1-\underbrace{p_2}_{\cancel{p_2-p_1u}+p_1})^\nu$$

$$= \frac{i}{16\pi^2}\int_0^\infty \frac{ds\,s}{s^2} \int_0^1 du\; p_1^\nu(1-u)\, e^{-ism^2(1-u)^2}$$

$$= -\frac{i}{16\pi^2} p_1^\nu \int_0^1 du\,(1-u)\ln(m^2(1-u)^2 s_0)$$

$$= -\frac{i}{16\pi^2} p_1^\nu \left[\frac{1}{2}\ln m^2 s_0 + \underbrace{2\int_0^1 du \ln(1-u)(1-u)}_{\int_0^1 du\, u\ln u = \left(\frac{1}{2}u^2\ln u - \frac{u^2}{4}\right)\Big|_0^1 = -\frac{1}{4}}\right]$$

$$= \frac{-i}{32\pi^2} p_1^\nu [\ln m^2 s_0 - 1]$$

$$\int \frac{(dp_2)}{(2\pi)^4} \frac{1}{p_2'^2+m^2} \frac{1}{(p_1'^2-p_2')^2} \underbrace{(p_1-p_2)^\nu}_{(p_2'-p_1')^\nu}$$

$$= -\int_0^\infty ds\, s \int_0^1 du \int \frac{(dp_2')}{(2\pi)^4} e^{-is[(p_2'-p_1'u)^2 + m^2(1-u)^2]} (p_2'-p_1')^\nu$$

$$= +\frac{i}{32\pi^2} p_1'^\nu [\ln m^2 s_0 - 1]$$

$$\therefore \quad (A - 4m^2C)(p_1-p_1')^\nu = (p_1-p_1')^\nu \left\{\frac{-i}{32\pi^2}\right\}\Big[\ln m^2 s_0 - \frac{1}{2}(\ln m^2 s_0 - 1)\Big]$$

$$= \frac{-i}{32\pi^2}(p_1-p_1')^\nu \frac{1}{2}[\ln m^2 s_0 + 1]$$

while from p. 16

$$(A+Bk^2) + 3A - 4m^2C = \frac{-i}{16\pi^2}\ln m^2 s_0 \qquad (1)$$

$$A - 4m^2C = \frac{-i}{64\pi^2}[\ln m^2 s_0 + 1] \qquad (2)$$

p. 14: $$A + Bk^2 = \frac{1}{2}\frac{-i}{32\pi^2}[\ln m^2 s_0 - 1] \qquad (3)$$

$$3\,(2) + (3) - (1) = -\frac{i}{16\pi^2}\Big[\ln m^2 s_0 \left(+\frac{3}{4}+\frac{1}{4}-1\right) + \left(+\frac{3}{4} - \frac{1}{4}\right)\Big] = -\frac{i}{32\pi^2}$$

$$= -8m^2C$$

Redo integrations using dim. reg. (Eucl. s, [illegible])

$$\int \frac{d^d p}{(2\pi)^d}\, e^{-sp^2} = \frac{1}{2^d \pi^{d/2} s^{d/2}}$$

so $\displaystyle\int_0^\infty \frac{ds\, s}{s^{d/2}}\, e^{-m^2 s} = (m^2)^{\frac{d}{2}-2}\, \Gamma(2-d/2)$ ← Pole at $d=4$

$$\Rightarrow a = -\frac{i}{64\pi^2 m^2}\,(m^2)^{\frac{d}{2}-2}\,\Gamma(2-d/2)\Big[1-\underbrace{\int_0^1 du\,(1-u)^{d-4}}_{\frac{1}{d-3}}\Big]$$
(p16)

$$= -\frac{i}{64\pi^2 m^2}\,\Gamma(2-d/2)\,\frac{\overbrace{d-4}^{-2(2-d/2)}}{d-3}\Bigg|_{d=4} \qquad d\simeq 4$$

$$= -\frac{i}{64\pi^2 m^2}(-2)\,\Gamma(3-d/2)\Big|_{d=4} = \frac{i}{32\pi^2 m^2}$$ agrees w/ p. 16

$$4A + Bk^2 - 4m^2 C = +\frac{i}{16\pi^2}\,(m^2)^{\frac{d}{2}-2}\,\Gamma(2-d/2)$$

$$A - 4m^2 C = \frac{i}{16\pi^2}\Big\{\frac{1}{2}(m^2)^{\frac{d}{2}-2}\,\Gamma(2-d/2) - \frac{1}{2}(m^2)^{\frac{d}{2}-2}\,\Gamma(2-d/2)\int_0^1 du\,(1-u)(1-u)^{d-4}\Big\}$$

$$= \frac{i}{32\pi^2}(m^2)^{\frac{d}{2}-2}\,\Gamma(2-d/2)\Big\{1 - \frac{1}{d-2}\Big\}$$

$$A + Bk^2 = \frac{i}{16\pi^2}\,\frac{1}{2}(m^2)^{d/2-2}\,\Gamma(2-d/2)\Big\{\frac{1}{d-2}\Big\}$$ (same integral as

$$3(A-4m^2C) - (4A + Bk^2 - 4m^2 C) + A + Bk^2 = -8m^2 C =$$

$$= \frac{i}{16\pi^2}\,\Gamma(2-d/2)\Big[-1 + \frac{3}{2}\Big(1-\frac{1}{d-2}\Big) + \frac{1}{2}\,\frac{1}{d-2}\Big]$$

$$= \frac{i}{16\pi^2}\,\Gamma\Big(2-\frac{d}{2}\Big)\,\frac{d-4}{2(d-2)} = -\frac{i}{16\pi^2}\,\frac{1}{2} = -\frac{i}{32\pi^2}$$ agrees w/ p. 18.

But this is not quite right, because in the trace on p. 14, in d dimensions, $4A \to dA$. This means, on the bottom of p. 14, we should have

$$(d-1)(A-4m^2C) - (dA + Bk^2 - 4m^2C) + A + Bk^2$$

$$= -(d-2)\,4m^2 C$$

$$= \frac{i}{16\pi^2}\,\Gamma(2-d/2)\underbrace{\left[-1 + \frac{d-1}{2}\left(1-\frac{1}{d-2}\right) + \frac{1}{2}\frac{1}{d-2}\right]}_{\frac{1}{d-2}\left(-\frac{1}{2}+\frac{1}{2}\right) - 1 + \frac{d-1}{2} - \frac{1}{2} \;=\; \frac{d}{2} - 2}$$

$$= -\frac{i}{16\pi^2}, \quad C = +\frac{1}{8m^2}\frac{i}{16\pi^2} \quad d=4$$

This shows 1) you must be consistent!
2) the unregulated approach gives the wrong answer.

Now insert this back into p12:

$$4ma(p_1-p_1')^\lambda + 4(p_1-p_1')^\lambda a\gamma(p_1-p_1')$$
$$-4C(p_1-p_1')^\lambda \underbrace{\gamma(p_1-p_1')}_{-2m} - 4Bk^\lambda \underbrace{\gamma(p_1+p_1')}_{0}$$

$$= 4m(p_1-p_1')^\lambda\left[a-2a+2C\right]$$
$$= 4m(p_1-p_1')^\lambda(-a+2C)$$

and then $(p_1-p_1')^\lambda = -\frac{1}{2}\{\gamma^\lambda, \gamma(p_1-p_1')\}$

$$= -\frac{1}{2}\gamma^\lambda\left[\gamma(p_1+p_1') - 2\gamma p_1'\right]$$
$$-\frac{1}{2}\left[\gamma(-(p_1+p_1')) + 2\gamma p_1\right]\gamma^\lambda$$
$$= 2m\gamma^\lambda + \frac{1}{2}[\gamma k, \gamma^\lambda]$$
$$= 2m\gamma^\lambda + \frac{1}{2}(-i)2\sigma^{\mu\lambda}k_\mu$$
$$= 2m\gamma^\lambda + \underbrace{i\sigma^{\lambda\nu}k_\nu}_{\text{magnetic moment term}}$$

So returning to the amplitude on p. 10:

$$-e^3\bar{\psi}(-p_1)\, i\sigma^{\lambda\nu}\underbrace{k_\nu A_\lambda(k)}_{\frac{1}{2}(k_\nu A_\lambda - k_\lambda A_\nu) = \frac{1}{2i}F_{\nu\lambda}(k)}\, \psi(-p_1')\, 4m(2C-a)$$

$$\Rightarrow +e^3\int(dx)\,\bar{\psi}(x)\sigma^{\mu\nu}\frac{1}{2}F_{\mu\nu}\,\psi(x)\,4m(2C-a)$$

$$= -i\int(dx)\,\bar{\psi}(x)\frac{e}{2m}\sigma F\,\psi(x)\frac{g-2}{2}, \qquad \frac{g-2}{2} = i4\pi\alpha\, 8m^2(2C-a) = i4\pi\alpha\left(+\frac{i}{8\pi^2}-\frac{i}{4\pi^2}\right)$$

$$\boxed{\frac{g-2}{2} = \alpha/2\pi}$$