Homework #6

6-1 For a null (light-like) vector $p^\mu = (\vec{p}, p^0)$

$$p^2 = 0 = -p^{0\,2} + \vec{p}^{\,2}$$

we can define a second null vector by

$$\bar{p}^\mu = (-\vec{p}, p^0), \quad \bar{p}^2 = 0$$

Now $(p+\bar{p})^\mu = (\vec{0}, 2p^0)$ is timelike,

$$(p+\bar{p})^2 = -4p^{0\,2} < 0$$

while $(p-\bar{p})^\mu = (2\vec{p}, 0)$ is spacelike,

$$(p-\bar{p})^2 = 4\vec{p}^{\,2} > 0.$$

Therefore $\dfrac{(p+\bar{p})^\mu (p+\bar{p})^\nu}{(p+\bar{p})^2}$ is a projection oper[ator] into the time direction; while

$\dfrac{(p-\bar{p})^\mu (p-\bar{p})^\nu}{(p+\bar{p})^2}$ is a projection operat[or] into the direction of $\vec{p}$. Span the two remain[ing] directions by vectors orthogonal to $\vec{p}$:

$$\vec{e}_1, \vec{e}_2 \; ; \quad \vec{p}\cdot\vec{e}_\lambda = 0$$

Introduce four-vectors: $e^\mu_{p\lambda} = (0, \vec{e}_\lambda)$

$$\therefore \; g^{\mu\nu} = \frac{(p+\bar{p})^\mu (p+\bar{p})^\nu}{(p+\bar{p})^2} + \frac{(p-\bar{p})^\mu (p-\bar{p})^\nu}{(p-\bar{p})^2} + \sum_{\lambda=1}^{2} e^\mu_{p\lambda} e^{\nu *}_{p\lambda}$$

Then $|\langle 0_+|0_-\rangle^J|^2 = e^{-\int (dx)(dx') J^\mu(x) \operatorname{Re} \frac{1}{i} D_+(x-x') J_\mu(x')}$

$$= e^{-\int d\tilde{p}\, J^\mu(-p) J_\mu(p)}$$

$$= e^{-\int d\tilde{p}\, J_\mu(-p) g^{\mu\nu} J_\nu(p)}$$

Now $\partial_\mu J^\mu(x) = 0 \Rightarrow p_\mu J^\mu(p) = 0$

so $J_\mu(-p) g^{\mu\nu} J_\nu(p)$

$$= J_\mu(-p)\left[\cancel{\frac{\bar{p}^\mu \bar{p}^\nu}{2p\bar{p}}} + \cancel{\frac{\bar{p}^\mu \bar{p}^\nu}{-2p\bar{p}}} + \sum_\lambda e^\mu_{p\lambda} e^{\nu *}_{p\lambda}\right] J_\nu(p)$$

$$= \sum_{\lambda=1}^{2} \left(J_\mu(-p) e^\mu_{p\lambda}\right)\left(e^{*\nu}_{p\lambda} J_\nu(p)\right)$$

$$= \sum_\lambda \left| e^{*\nu}_{p\lambda} J_\nu(p)\right|^2$$

and $|\langle 0_+|0_-\rangle^J|^2 = e^{-\sum_{p\lambda} |J_{p\lambda}|^2} \leq 1$

where $J_{p\lambda} = \sqrt{d\tilde{p}}\, e^{*\nu}{}_{p\lambda} J_\nu(p)$.

6-2 Recall that under a coordinate transformation a vector transforms as

$$\delta J_\mu = -\delta x^\nu \partial_\nu J_\mu - (\partial_\mu \delta x^\nu) J_\nu$$

A Lorentz transformation has

$$\delta x^\nu = \delta\omega^{\mu\nu} x_\mu$$ with $\delta\omega^{\mu\nu}$ constant parameters, so

$$\delta J_\mu = -\delta\omega^{\lambda\nu} x_\lambda \partial_\nu J_\mu - \delta\omega_\mu{}^\nu J_\nu$$

or $$\delta J^\mu = -\delta\omega^{\lambda\nu} x_\lambda \partial_\nu J^\mu - \delta\omega^{\mu\nu} J_\nu$$

Rewrite this as

because of AS of $\delta\omega^{\alpha\beta}$

$$\delta J^\mu = -\frac{i}{2}\delta\omega^{\alpha\beta}\Big[\big(x_\alpha \tfrac{1}{i}\partial_\beta - x_\beta \tfrac{1}{i}\partial_\alpha\big) J^\mu + \tfrac{1}{i}\big(\delta^\mu_\alpha g_{\beta\lambda} - \delta^\mu_\beta g_{\alpha\lambda}\big) J^\lambda\Big]$$

$$= -\frac{i}{2}\delta\omega^{\alpha\beta}\left[L_{\alpha\beta} + S_{\alpha\beta}\right]^\mu{}_\lambda J^\lambda$$

where $L_{\alpha\beta}$ is a diagonal matrix (orbital angular momentum)

$$(L_{\alpha\beta})^\mu{}_\lambda = \delta^\mu_\lambda \left(x_\alpha \frac{1}{i}\partial_\beta - x_\beta \frac{1}{i}\partial_\alpha\right)$$

and the spin angular momentum is

$$(S_{\alpha\beta})^\mu{}_\lambda = \frac{1}{i}\left(\delta^\mu_\alpha g_{\beta\lambda} - \delta^\mu_\beta g_{\alpha\lambda}\right)$$

For example
$$(S_{12})^1{}_2 = -i = -(S_{12})^2{}_1$$
while $(S_{03})^0{}_3 = -i = +(S_{03})^3{}_0 \quad (g_{00} = -1)$

So S_{kl} are antisymmetrical and Hermitian, while S_{0k} are symmetrical and skew Hermitian

We pass to Euclidean space by letting
$$x_4 = ix^0$$

In analogy, $\quad S_{k4} = iS_k{}^0 = -iS_{k0} = iS_{0k}$

and $\quad (S_{\mu\nu})_{4k} = i(S_{\mu\nu})^0{}_k$

so the only nonzero elements of S_{4k} are
$$(S_{4k})_{4k} = i(S_{4k})^0{}_k = i(-iS_{0k})^0{}_k$$
$$= \frac{1}{i}$$
$$(S_{4k})_{k4} = i(S_{4k})_k{}^0 = i(iS_{k0})_k{}^0 = -g_{kk}\delta^0_0\frac{1}{i}$$
$$= -\frac{1}{i}$$
$$\therefore (S^E_{\mu\nu})_{\lambda\kappa} = \frac{1}{i}(\delta_{\lambda\mu}\delta_{\nu\kappa} - \delta_{\lambda\nu}\delta_{\mu\kappa})$$
which are explicitly antisymmetrical and Hermitian.
There is no distinction between the 4th dim. & the others

6-3 For spin $\frac{1}{2}$ the spin matrix is

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

or

$$\sigma_{kl} = \frac{i}{2}[\gamma_k, \gamma_l] = i\gamma_k\gamma_l, \quad k \neq l$$

$$\sigma^0{}_k = \frac{i}{2}[\gamma^0, \gamma_k] = i\gamma^0\gamma_k$$

$$\sigma_{kl}^+ = -i\gamma_l^+\gamma_k^+ = -i\gamma_l\gamma_k = i\gamma_k\gamma_l = \sigma_{kl}$$

$$\sigma_{kl}^T = i\gamma_l^T\gamma_k^T = i\gamma_l\gamma_k = -i\gamma_k\gamma_l = -\sigma_{kl}$$

σ_{kl} is antisymmetric and Hermitian.

$$(\sigma^0{}_k)^+ = -i\gamma_k^+\gamma^{0+} = +i\gamma_k\gamma^0 = -i\gamma^0\gamma_k = -\sigma^0{}_k$$

$$(\sigma^0{}_k)^T = i\gamma_k^T\gamma^{0T} = -i\gamma_k\gamma^0 = +i\gamma^0\gamma_k = \sigma^0{}_k$$

$\sigma^0{}_k$ is symmetric + skew Hermitian

The Euclidean transcription (see prob. 6-2) is

$$\sigma_{k4} = i\sigma_k{}^0 = -i\sigma^0{}_k = \gamma^0\gamma_k$$

is now symmetric and Hermitian. Thus, there's a difference between the 4th direction + the other three. Could we make a unitary transform on σ_{k4} to make it antisymmetric & Hermitian? We must do so without changing σ_{kl}.

That is, consider $U\sigma_{k4}U^{-1}$ where $U = e^{iH\alpha}$
H = Hermitian matrix. Require $U\sigma_{kl}U^{-1} = \sigma_{kl}$ and
$U\sigma_{kl}U^{-1}$ antisymmetric + Hermitian ⇒ Imaginary. The
only possibilities are $H = \gamma^0, i\gamma_5, \gamma^0\gamma_5$ which
are Hermitian and all commute with σ_{kl}. But these
are all imaginary (Hermitian + antisymmetric) matrices, hence U is real and so
is $U\sigma_{kl}U^{-1}$. [The other Hermitian matrices are $\gamma^0\gamma^\mu$, $\gamma^0\sigma^{\mu\nu}$
and $i\gamma_5\gamma^0\gamma^\mu$; these do not commute with σ_{kl} because
these are, respectively, vectors, tensors, + vectors under
rotations. γ^0, $i\gamma_5$, + $\gamma^0\gamma_5$ are scalars.]

But if fermions have charge, represented by
an additional index on which

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad q^+ = q$$

can act, we can accomplish this by taking
$U = e^{i\alpha q\gamma^0}$. For U obviously commutes w/ σ_{kl}, +

$$U\sigma_{k4}U^{-1} = e^{i\alpha q\gamma^0} \underbrace{\sigma_{k4}}_{\gamma^0\gamma_k} \underbrace{e^{-i\alpha q\gamma^0}}_{\cos\alpha - iq\gamma^0\sin\alpha}$$

$$= e^{i\alpha q\gamma^0} e^{i\alpha q\gamma^0}\gamma^0\gamma_k = e^{2i\alpha q\gamma^0}\gamma^0\gamma_k$$

$$= (\cos 2\alpha + iq\gamma^0 \sin 2\alpha)\gamma^0\gamma_k$$

$\gamma^0\gamma_k$ is symmetrical, but $iq\gamma_k$ is antisymmetrical
and Hermitian! The problem is solved if we take
$\alpha = \pi/4$ ⇒ $(\sigma_{k4})_E = iq\gamma_k$ $(\sigma_{kl})_E = i\gamma_k\gamma_l$ $(k \neq l)$