Homework #5

5-1. Because $\int_0^\infty \frac{dt}{t} t^\alpha e^{-t} = \Gamma(\alpha) \quad (\alpha > 0)$

we can write

$$\int_0^\infty \frac{ds}{s} s^{n+1} e^{-s(\ell^2+m^2)} = \frac{1}{(\ell^2+m^2)^{n+1}} \int_0^\infty \frac{dt}{t} t^{n+1} e^{-t}$$

$$= \frac{1}{(\ell^2+m^2)^{n+1}} \Gamma(n+1)$$

Now $\int_{-\infty}^\infty dx\, e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$, so

$$\int_{-\infty}^\infty \frac{d^d\ell}{(2\pi)^d} e^{-s\ell^2} = \frac{1}{(2\pi)^d}\left[\int_{-\infty}^\infty d\ell\, e^{-s\ell^2}\right]^d$$

$$(\ell^2 = \ell_1^2 + \ell_2^2 + \ell_3^2 + \cdots + \ell_d^2)$$

$$= \frac{1}{(2\pi)^d}\left(\frac{\pi}{s}\right)^{d/2} = \frac{1}{2^d \pi^{d/2} s^{d/2}}$$

Therefore $\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2+m^2)^{n+1}} = \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{ds}{s} s^{n+1} e^{-s\ell^2} e^{-sm^2}$

$$= \int_0^\infty \frac{ds}{s} s^{n+1} e^{-sm^2} \frac{1}{\Gamma(n+1)} \frac{1}{2^d \pi^{d/2} s^{d/2}}$$

$$= \frac{1}{2^d \pi^{d/2}} \frac{1}{\Gamma(n+1)} \int_0^\infty \frac{ds}{s} s^{n+1-d/2} e^{-sm^2}$$

$$= \frac{1}{2^d \pi^{d/2}} (m^2)^{d/2-1-n} \frac{\Gamma(n+1-d/2)}{\Gamma(n+1)}$$

where the latter step is only valid if

$$n+1-d/2 > 0$$

Nevertheless, we will take the RHS of this equation to define (analytically continue) the LHS for all d + n.

5-2 $d=4,\ n=0$: $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2+m^2}$ is quadratically divergent. The analytically continued result for $n=0$ is

$$\frac{1}{2^d \pi^{d/2}}\, m^{d-2}\, \frac{\Gamma(1-d/2)}{\Gamma(1)} ;$$

$\Gamma(1-d/2)$ has simple poles at $d=2,4,6,8,\ldots$ since $\Gamma(x)$ has simple poles at $x=0,-1,-2,-3,\ldots$

$d=4,\ n=1$: $\displaystyle\int \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{(\ell^2+m^2)^2}$ is logarithmically divergent. The analytically continued result for $n=1$ is

$$\frac{1}{2^d \pi^{d/2}}\, m^{d-4}\, \frac{\Gamma(2-d/2)}{\Gamma(2)}$$

$\Gamma(2-d/2)$ has simple poles at $d=4,6,8,\ldots$

5-3

$$= \frac{-i\lambda}{2}\int \frac{(d\ell)}{(2\pi)^4}\, \frac{-i}{(\ell^2+m^2)}$$

Go to Euclidean space : $\int (dx) \to -i\int (dx)_E \qquad \Delta_+ \to i\Delta_E$

$$\longrightarrow -\frac{\lambda}{2}\int \frac{(d\ell)_E}{(2\pi)^4}\, \frac{1}{(\ell_E^2+m^2)}$$

Proceed by dimensional regularization:

$\lambda \to \lambda \mu^{4-d}$ so above (prob. 5-2)

$$\to -\frac{1}{2}\lambda \mu^{4-d} \frac{1}{2^d \pi^{d/2}} m^{d-2} \frac{\Gamma(1-d/2)}{\Gamma(1)}$$

Let $d = 4+\epsilon$, $\epsilon \ll 1$:

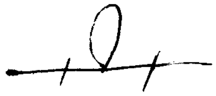
$$= -\frac{1}{2}\lambda \mu^{-\epsilon} \frac{1}{2^{4+\epsilon}\pi^{2+\epsilon/2}} m^{2+\epsilon}\,\Gamma(-1-\epsilon/2)$$

Now $\Gamma(-1-\epsilon/2) = -\dfrac{1}{1+\epsilon/2}\Gamma(-\epsilon/2)$

$$= \frac{1}{1+\epsilon/2}\frac{1}{\epsilon/2}\Gamma(1-\epsilon/2)$$

$$= \frac{2}{\epsilon}\left(1-\frac{\epsilon}{2}\right)\left[\Gamma(1) - \frac{\epsilon}{2}\Gamma'(1)\right]$$

$$= \frac{2}{\epsilon} - 1 - \Gamma'(1)$$



$$= -\frac{\lambda}{2}\frac{m^2}{16\pi^2}\underbrace{\left(\frac{m}{\mu\, 2\sqrt{\pi}}\right)^{\epsilon}}_{1+\frac{\epsilon}{2}\ln\frac{m^2}{4\pi\mu^2}}\left[\frac{2}{\epsilon} - 1 - \Gamma'(1)\right]$$

$$\boxed{\begin{aligned}\psi(1+x) &= \frac{d}{dx}\ln\Gamma(1+x)\\ &= \frac{d}{dx}\left[\ln\Gamma(x) + \ln x\right]\\ &= \psi(x) + \frac{1}{x}\end{aligned}}$$

$$= -\frac{\lambda m^2}{32\pi^2}\left\{\frac{2}{\epsilon} + \ln\frac{m^2}{4\pi\mu^2} - 1 - \underbrace{\Gamma'(1)}_{\Gamma(1)\psi(1)}\right]$$

$$= -\frac{\lambda m^2}{32\pi^2}\left\{\frac{2}{\epsilon} + \ln\frac{m^2}{4\pi\mu^2} - \psi(2)\right\} = \Sigma$$

To order λ, $m_0^2 = m_R^2 + \delta m^2$

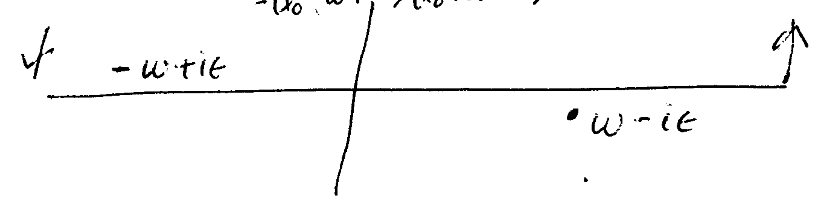
$$m_R^2 = m_0^2 - \Sigma \,, \quad \delta m^2 = -\Sigma = \frac{\lambda m^2}{32\pi^2}\left\{\frac{2}{\epsilon} + \underset{\uparrow}{\text{finite}}\right\}$$

<u>5-4</u> Consider

$$\int \frac{(d\ell)}{(2\pi)^4} \frac{1}{(\ell^2 + m^2 - i\epsilon)^3}$$

Do ℓ_0 integral first

$$\int_{-\infty}^{\infty} \frac{d\ell_0}{2\pi} \frac{1}{\underbrace{(\ell_0^2 + \omega^2 - i\epsilon)}_{-(\ell_0 - \omega + i\epsilon)(\ell_0 + \omega - i\epsilon)}{}^3} \qquad \omega^2 = \vec{\ell}^{\,2} + m^2$$



Close in UHP, say,

$$-\int_{-\infty}^{\infty} \frac{d\ell_0}{2\pi} \frac{i}{(\ell_0 - \omega + i\epsilon)^3 (\ell_0 + \omega - i\epsilon)^3}$$

$$= -i\, \underbrace{\frac{1}{2}\left(\frac{d}{d\ell_0}\right)^2 \frac{1}{(\ell_0 - \omega + i\epsilon)^3}}_{12 \quad \frac{1}{(-2\omega)^5}} \Bigg|_{\ell_0 = -\omega + i\epsilon} = \frac{i\,3}{2\cdot 8\,\omega^5}$$

Now use result of problem 5-1:

$$\int \frac{d^3 l}{(2\pi)^3} \frac{3i}{16\,\omega^5} = \int \frac{d^3 l}{(2\pi)^3} \frac{3i}{16(\vec{l}^{\,2}+m^2)^{5/2}}$$

$(d=3,\ n=3/2)$

$$= \frac{m^{3-3-2}}{2^3 \pi^{3/2}} \frac{\Gamma(\frac{5}{2}-3/2)}{\Gamma(5/2)} \frac{3i}{16}$$

$$= \frac{\cancel{3}i}{16} \frac{m^{-2}}{\cancel{8}_{2}\,\pi^{3/2}} \frac{1}{\frac{\cancel{3}}{\cancel{2}}\cdot\frac{1}{\cancel{2}}\sqrt{\pi}} = \frac{i}{32 m^2 \pi^2}$$

Directly, using dim-reg: $\quad l_0 \to i l_4$

$$\int \frac{(dl)}{(2\pi)^4} \frac{1}{(l^2+m^2-i\epsilon)^3} = i \int \frac{(dl)_E}{(2\pi)^4} \frac{1}{(l_E^2+m^2)^3}$$

$(d=4,\ n=2)$

$$= i\,\frac{m^{4-4-2}}{2^4 \pi^2} \frac{\Gamma(3-2)}{\Gamma(3)} = i \frac{1}{16\pi^2 m^2} \frac{1}{2} = i \frac{1}{32\pi^2 m^2}$$