4-1

Homework # 4

$$\mathcal{L} = -\frac{1}{2}\psi\gamma^0\left[\gamma\frac{1}{i}\partial + m\right]\psi,$$

$$\delta\mathcal{L} = -\frac{1}{2}\delta\psi\,\gamma^0\left(\gamma\frac{1}{i}\partial + m\right)\psi - \frac{1}{2}\psi\gamma^0\left(\gamma\frac{1}{i}\partial + m\right)\delta\psi$$

$$\delta\psi = \delta x^\nu\partial_\nu\psi + \frac{i}{4}\sigma^{\mu\nu}\psi\,\partial_\mu\delta x_\nu$$

$$\therefore -\delta\mathcal{L} = \delta x^\nu\partial_\nu\mathcal{L} - \frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{i}(\partial_\mu\delta x^\nu)\partial_\nu\psi$$

$$-\frac{1}{2}\psi\gamma^0\left[\gamma^\lambda, \frac{i}{4}\sigma^{\mu\nu}\right]\partial_\lambda\psi\;\partial_\mu\delta x_\nu$$

$$-\frac{1}{2}\psi\gamma^0\gamma^\lambda\frac{i}{4}\sigma^{\mu\nu}\psi\;\partial_\lambda\partial_\mu\delta x_\nu$$

since, again, $\sigma^{\mu\nu\dagger}\gamma^0 = \gamma^0\sigma^{\mu\nu}$. Now

$$[\gamma^\lambda, \sigma^{\mu\nu}] = 2i(g^{\lambda\nu}\gamma^\mu - g^{\lambda\mu}\gamma^\nu)$$

and $\gamma^\lambda\sigma^{\mu\nu} = \gamma^\lambda \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$= \frac{i}{2}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu)$$

$$(\partial_\lambda\partial_\mu \text{ sym.}) \rightarrow \frac{i}{2}\left[\frac{1}{2}\{\gamma^\lambda, \gamma^\mu\}\gamma^\nu - \frac{1}{2}(\gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda)\right]$$

$$= -\frac{i}{2}g^{\lambda\mu}\gamma^\nu - \frac{i}{4}\{-\gamma^\lambda 2g^{\mu\nu} + \gamma^\nu 2g^{\mu\lambda} - \gamma^\mu 2g^{\lambda\nu}\}$$

$$= -ig^{\lambda\mu}\gamma^\nu + \frac{i}{2}\gamma^\lambda g^{\mu\nu} + \frac{i}{2}\gamma^\mu g^{\lambda\nu}$$

$$\text{So } -\delta\mathcal{L} = \delta x^\nu\partial_\nu\mathcal{L} - \partial_\mu\delta x_\nu\left[\frac{1}{2}\psi\gamma^0\gamma^\mu\frac{1}{i}\partial^\nu\psi - \frac{1}{4}\left(\psi\gamma^0\gamma^\mu\frac{1}{i}\partial^\nu\psi - \psi\gamma^0\gamma^\nu\frac{1}{i}\partial^\mu\psi\right)\right]$$

$$+ \frac{1}{8}\psi\gamma^0\gamma^\alpha\psi\left[\partial^2\delta x_\alpha - \partial_\alpha\partial_\mu\delta x^\mu\right]$$

Now $\gamma^0\gamma^\alpha$ is a symmetrical matrix, while $\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha$ (classical fermion fields anti-commute — Grassmann elements — as we'll see later). So the last term here is zero,

and $$-\delta\mathcal{L} = \partial_\nu[\delta x^\nu \mathcal{L}] - (\partial_\nu \delta x^\nu)\mathcal{L}$$
$$- \partial_\mu \delta x_\nu \left[\tfrac{1}{4}\psi\gamma^0(\gamma^\mu \tfrac{1}{i}\partial^\nu + \gamma^\nu \tfrac{1}{i}\partial^\mu)\psi\right]$$
$$= \partial_\nu(\delta x^\nu \mathcal{L}) - t^{\mu\nu}\partial_\mu \delta x_\nu$$

where $$\boxed{t^{\mu\nu} = \tfrac{1}{4}\psi\gamma^0(\gamma^\mu \tfrac{1}{i}\partial^\nu + \gamma^\nu \tfrac{1}{i}\partial^\mu)\psi + g^{\mu\nu}\mathcal{L}}$$

Note that $t^{\mu\nu} = t^{\nu\mu}$.

Now the action principle says

$$0 = \delta W = \int \delta\mathcal{L} = -\int (dx)\,[\partial_\nu(\delta x^\nu \mathcal{L}) - t^{\mu\nu}\partial_\mu \delta x_\nu]$$

so integrating by parts and omitting surface integrals at ~~spatial~~ infinity,

$$0 = \int (dx)\, \delta x_\nu (\partial_\mu t^{\mu\nu}),$$

so since δx_ν is arbitrary $\partial_\mu t^{\mu\nu} = 0$. [This, of course, can be verified directly from the Dirac eqn.]

4.2 Take the trace: $t = t^{\mu}_{\ \mu} = \partial^{\mu}\varphi\partial_{\mu}\varphi + 4\mathcal{L}$

$$= \partial^{\mu}\varphi\partial_{\mu}\varphi + 4\left(-\tfrac{1}{2}\right)\left(\partial^{\mu}\varphi\partial_{\mu}\varphi + m^2\varphi^2\right)$$

$$= -\partial^{\mu}\varphi\partial_{\mu}\varphi - 2m^2\varphi^2 \rightarrow -\partial^{\mu}\varphi\partial_{\mu}\varphi$$

as $m \rightarrow 0$. Thus the scale current $x_{\mu}t^{\mu\nu}$ is not conserved.

Suppose we take

$$\Theta^{\mu\nu} = t^{\mu\nu} + a\left(\partial^{\mu}\partial^{\nu} - g^{\mu\nu}\partial^2\right)\varphi^2$$

and require $\Theta^{\mu}_{\ \mu} \rightarrow 0$ as $m \rightarrow 0$.

$$\Theta^{\mu}{}_{\mu} = -\partial_\mu\varphi\partial^\mu\varphi - 2m^2\varphi^2$$
$$+ a(-3)\partial^2\varphi^2$$

Now $\partial^2\varphi^2 = \partial_\lambda\partial^\lambda\varphi^2 = 2\partial_\lambda(\varphi\partial^\lambda\varphi)$
$$= 2\partial_\lambda\varphi\partial^\lambda\varphi + 2\varphi\partial^2\varphi$$

So $\Theta^{\mu}{}_{\mu} = -\partial_\mu\varphi\partial^\mu\varphi - 2m^2\varphi^2$
$$-6a(\partial_\mu\varphi\partial^\mu\varphi + \varphi\partial^2\varphi)$$

The $\partial_\mu\varphi\partial^\mu\varphi$ terms cancel if $a = -\frac{1}{6}$,
and then
$$\Theta^{\mu}{}_{\mu} = \overbrace{-2m^2\varphi^2 + \varphi^2 m^2}^{-m^2\varphi^2} \to 0 \text{ as } m \to 0$$
which uses the Klein-Gordon eqn.

$$\therefore \Theta^{\mu\nu} = t^{\mu\nu} - \tfrac{1}{6}(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)\varphi^2$$

$\theta^{\mu\nu}$ & $t^{\mu\nu}$ are equally good at describing the flow of energy & momentum; both are conserved, since
$$\partial_\mu(\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2) = \partial^\nu\partial^2 - \partial^\nu\partial^2 \equiv 0;$$

and $\mathcal{E} = \int (d\vec{x})\, t^{00} = \int (d\vec{x})\, \Theta^{00}$

because

$$(\partial^0 \partial^0 - g^{00} \partial^2)\varphi^2 = [\partial^{0\,2} + (-\partial^{0\,2} + \nabla^2)]\varphi^2$$
$$= \nabla^2 \varphi^2$$

and $\int (d\vec{x})\, \nabla^2 \varphi^2 = \oint_\Sigma d\vec{\sigma} \cdot \vec{\nabla} \varphi^2$

where Σ is the surface bounding the volume, which will vanish as $\Sigma \to \infty$ for bounded field distributions. Similarly

$\vec{P}_i = \int (d\vec{x})\, t^{0i} = \int (d\vec{x})\, \Theta^{0i}$

because $\int (d\vec{x})\, \partial^0 \partial^i \varphi^2 = \oint_\Sigma d\sigma_i\, \partial_0 \varphi^2 \to 0$

as $\Sigma \to \infty$ for bounded field distributions.

$\Theta^{\mu\nu}$ is preferred

1) if one is discussing scale invariance
2) if one is coupling scalar particles to gravity.

More generally, one has the freedom to add to $t^{\mu\nu}$ a term $\partial_\kappa \partial_\lambda m^{\mu\nu,\kappa\lambda}$ where $m^{\mu\nu,\kappa\lambda} = m^{\nu\mu,\kappa\lambda} = m^{\mu\nu,\lambda\kappa}$

and $m^{\mu\nu,\kappa\lambda} + m^{\kappa\nu,\lambda\mu} + m^{\lambda\nu,\mu\kappa} = 0$

[In the above $m^{\mu\nu,\kappa\lambda} = (g^{\mu\kappa} g^{\nu\lambda} + g^{\nu\kappa} g^{\mu\lambda} - 2 g^{\mu\nu} g^{\kappa\lambda})(-\frac{1}{12})$]

This added term is identically conserved, $\partial_\mu \partial_\kappa \partial_\lambda m^{\mu\nu,\kappa\lambda} = 0$ by virtue of the cyclic property, and if $d\sigma_\mu$ is a 3-dimensional time-like surface element

$$0 = \int_\sigma d\sigma_\mu \partial_\kappa \partial_\lambda (m^{\mu\nu,\kappa\lambda} + m^{\kappa\nu,\lambda\mu} + m^{\lambda\nu,\mu\kappa})$$

$$= \int_\sigma d\sigma_\mu [\partial_\kappa \partial_\lambda m^{\mu\nu,\kappa\lambda} + 2 \partial_\kappa \partial_\lambda m^{\kappa\nu,\lambda\mu}]$$

$$= 3 \int_\sigma d\sigma_\mu \partial_\kappa \partial_\lambda m^{\mu\nu,\kappa\lambda}$$

$$- 2 \int_\sigma (d\sigma_\mu \partial_\kappa - d\sigma_\kappa \partial_\mu) \partial_\lambda m^{\mu\nu,\kappa\lambda}$$

or

$$\int d\sigma_\mu \partial_\kappa \partial_\lambda m^{\mu\nu,\kappa\lambda} = \frac{2}{3}\int_\sigma (d\sigma_\mu \partial_\kappa - d\sigma_\kappa \partial_\mu)\partial_\lambda m^{\mu\nu,\kappa\lambda}$$

$$= 0$$

since $\int_\sigma (d\sigma_\mu \partial_\nu - d\sigma_\nu \partial_\mu) f = 0$ for a system that is closed in space-like directions.

Thus $\int_\sigma d\sigma_\mu t^{\mu\nu} = \int_\sigma d\sigma_\mu \theta^{\mu\nu}$

We now can perform the Euclidean space transformation:

$$i|x^0 - x'|^0 \to |x_4 - x_4'|$$

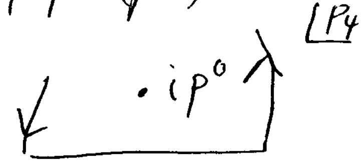
which gives

$$\Delta_+(x-x') \to i\,\Delta_E(x-x'),$$

$$\Delta_E(x-x') = \int \frac{(d\vec{p})}{(2\pi)^3}\,\frac{1}{2p^0}\,e^{i\vec{p}\cdot(\vec{x}-\vec{x}') - p^0|x_4-x_4'|}$$

But now we use the identity

$$\frac{1}{2p^0}\,e^{-p^0|x-x_4'|} = \int_{-\infty}^{\infty}\frac{dp_4}{2\pi}\,\frac{e^{ip_4(x_4-x_4')}}{p_4^2+(p^0)^2}$$



[The latter is, if $x_4 - x_4' > 0$

$$\frac{1}{2\pi}\,2\pi i\,\frac{e^{-p^0(x_4-x_4')}}{2ip^0} = \frac{1}{2p^0}\,e^{-p^0(x_4-x_4')} \qquad -ip^0$$

& if $x_4 - x_4' < 0$: $\quad -\frac{1}{2\pi}\,2\pi i\,\frac{1}{-2ip^0}\,e^{p^0(x_4-x_4')} = \frac{1}{2p^0}\,e^{-p^0|x_4-x_4'|}$

$$\therefore \Delta_E(x-x') = \int \frac{(d\vec{p})dp_4}{(2\pi)^4} \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x}')+ip_4(x_4-x_4')}}{p_4^2+\vec{p}^{\,2}+m^2}$$

$$= \int \frac{(dp)_E}{(2\pi)^4} \frac{e^{ip\cdot(x-x')_E}}{p_E^2+m^2}$$

as stated

<u>44</u> Of course, we can rotate back,

$$|x_4 - x_4'| \to i|x^0 - x'^0|,$$

to recover Δ_+, but, we can also keep rotating in the same sense:

$$|x_4 - x_4'| \to -i|x^0 - x'^0|$$

$$\Delta_E(x-x') \to \int \frac{(d\vec{p})}{(2\pi)^3} \frac{1}{2p^0} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')+ip^0|x^0-x'^0|}$$

$$= \int \frac{(d\vec{p})}{(2\pi)^3} \frac{1}{2p^0} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')+ip^0|x^0-x'^0|}$$

$$= i\Delta_-(x-x')$$

where $\Delta_-(x-x') = \begin{cases} -i\int d\tilde{p}\, e^{-ip\cdot(x-x')}, & (x-x')^0>0 \\ -i\int d\tilde{p}\, e^{ip\cdot(x-x')}, & (x-x')^0<0 \end{cases}$

Since $\displaystyle\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{e^{ip_0(x-x')^0}}{-p_0^2+m^2+i\epsilon} = -\frac{i}{2\omega_p}e^{i\omega_p|x^0-x'^0|}$

we have

$$\Delta_-(x-x') = \int\frac{(dp)}{(2\pi)^4}\frac{e^{ip\cdot(x-x')}}{p^2+m^2+i\epsilon}$$

in Minkowski space.

4.5. It is particularly easy to get explicit limiting forms for the Euclidean propagation function. Let

$$|x-x'| = R,$$ and choose the (4-dim) coordinate system so that

$$\vec{x}-\vec{x}'=0,\quad x_4-x'_4 = R$$

Then

$$\Delta_E(R) = \int \frac{(d\vec{p})}{(2\pi)^3}\,\frac{1}{2p^0}\,e^{-p^0 R}$$

$$p^0 = \sqrt{\vec{p}^{\,2} + m^2}\,, \quad (d\vec{p}) = 4\pi p^2\,dp$$

$$= 4\pi p\, p^0\, dp^0$$

$$= 4\pi \sqrt{p^{02} - m^2}\; p^0\, dp^0$$

$$\Delta_E(R) = \frac{1}{4\pi^2}\int_m^\infty dp^0 \sqrt{p^{02}-m^2}\; e^{-p^0 R}$$

If $m \ll \frac{1}{R}$, $mR \ll 1$

$$\Delta_E(R) \sim \frac{1}{4\pi^2}\int_{m\to 0}^\infty dp^0\, p^0\, e^{-p^0 R}$$

$$= \frac{1}{4\pi^2 R^2}\cdot \underbrace{\int_0^\infty dx\, x\, e^{-x}}_{1} = \frac{1}{4\pi^2 R^2}$$

If $mR \gg 1$

$$\Delta_E(R) = \frac{1}{4\pi^2}\,\frac{1}{R^2}\int_{mR}^\infty dx\, \underbrace{\sqrt{x^2 - m^2R^2}}_{\sqrt{x-mR}\,\underbrace{\sqrt{x+mR}}_{\simeq\sqrt{2mR}}}\, e^{-x}$$

$u = x - mR$: $$= \frac{1}{4\pi^2}\,\frac{1}{R^2}\sqrt{2mR}\int_0^\infty du\, u^{1/2} e^{-u-mR} = \frac{\sqrt{2m}}{4\pi^2 R^{3/2}}\, e^{-mR}\,\Gamma\!\left(\tfrac{3}{2}\right)$$

$\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, so

$$mR \gg 1: \quad \Delta_E(R) \sim \frac{\sqrt{2m}}{(4\pi R)^{3/2}} e^{-mR}$$