Homework #3

3-1

$$W = \int (dx)\,(\partial_\mu \varphi^* \partial^\mu \varphi + m^2 \varphi^* \varphi)$$

$$\delta\varphi^* : \quad \delta W = \int (dx)\,(\partial_\mu \delta\varphi^* \, \partial^\mu \varphi + m^2 \delta\varphi^* \varphi)$$

$$= \int (dx)\,[-\delta\varphi^* \, \partial_\mu \partial^\mu \varphi + \delta\varphi^* m^2 \varphi]$$

where we've integrated by parts + omitted the surface term. Since $\delta\varphi^*$ is arbitrary, we must have

$$-\partial^2 \varphi + m^2 \varphi = (-\partial^2 + m^2)\varphi = 0.$$

Varying wrt φ just gives the complex conjugate of this equation.

$$(-\partial^2 + m^2)\varphi^* = 0.$$

Now if $\varphi \to e^{i\Lambda}\varphi$, $\varphi^* \to e^{-i\Lambda}\varphi$, $W \to W$ if Λ is const. But if $\Lambda = \Lambda(x)$

$$W \to \int (dx)\left[\partial_\mu (e^{-i\Lambda}\varphi^*)\partial^\mu (e^{i\Lambda}\varphi) + m^2 \varphi^* \varphi\right]$$

$$= W + \int (dx)\,[\varphi^*(-i\partial_\mu \Lambda)\partial^\mu \varphi + \partial_\mu \varphi^* i(\partial^\mu \Lambda)\varphi$$
$$+ \partial_\mu \Lambda \, \partial^\mu \Lambda \quad \varphi^* \varphi]$$

For an infinitesimal $\Lambda = \delta\Lambda$, we keep only first-order terms, +

$$\delta W = -\int (dx)\,\partial_\mu \delta\Lambda\,[\,i\varphi^* \partial^\mu \varphi - i\partial_\mu \varphi^* \varphi\,]$$

$$\Rightarrow \quad j^\mu = i[\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*]$$

which is conserved by virtue of the stationary action principle.

Now δW can be exactly cancelled if we add an interaction

$$W_{int} = e\int A_\mu [i\varphi^* \partial^\mu \varphi - i\partial^\mu \varphi^* \varphi](dx)$$

where under a gauge transformation,

$$\delta A_\mu = \partial_\mu \delta\lambda, \qquad e\delta\lambda = \delta\Lambda,$$

$$\delta_{GT} W_{int} = e\int(\partial_\mu \delta\lambda)\, j^\mu \quad \text{precisely cancels } \delta W.$$

But this is not the end of the story, because W_{int} is not invariant under phase transformations

$$\delta_{PT} W_{int} = e\int(dx)\, A_\mu [i\varphi^* \partial^\mu(ie\delta\lambda)\varphi - i(\partial^\mu(-ie\delta\lambda))\varphi^*\varphi]$$

$$= -e^2\int(dx)\, A_\mu\, (\partial^\mu \delta\lambda)\, 2\varphi^*\varphi$$

So we must add a second interaction term

$$W'_{int} = e^2\int(dx)\, A_\mu A^\mu \varphi^* \varphi$$

which under $\delta A_\mu = \partial_\mu \delta\lambda$ precisely cancels $\delta_{PT} W_{int}$, and which is phase invariant. Put this all together

$$W + W_{int} + W'_{int} = \int(dx)\{[\partial_\mu \varphi^* \partial^\mu \varphi^* + m^2 \varphi^* \varphi] + e[i\varphi^* \partial^\mu \varphi - i\partial^\mu \varphi^* \varphi]A_\mu + e^2 A^\mu A_\mu \varphi^* \varphi\}$$

$$= \int (dx) \left[(\partial_\mu + ieA_\mu)\varphi^* (\partial^\mu - ieA^\mu)\varphi + m^2 \varphi^* \varphi \right]$$

$$= \int (dx) \left(D_\mu^* \varphi^* D^\mu \varphi + m^2 \varphi^* \varphi \right)$$

$D_\mu = \partial_\mu - ieA_\mu$ is the gauge-covariant derivative

---

3-2 Under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$A_0 \to A_0 + \partial_0 \Lambda = A_0'$$

so if $\Lambda = \int_{-\infty}^{t} A^0(\vec{x}, t')\,dt'$

$$\partial_0 \Lambda = \frac{\partial}{\partial t} \Lambda = A^0(\vec{x}, t) = -A_0(\vec{x}, t)$$

so $A_0' = 0$.

On the other hand

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda = \vec{A} + \int_{-\infty}^{t} \vec{\nabla} A^0(\vec{x}, t')\,dt$$

$$\vec{\nabla} \cdot \vec{A} + \int_{-\infty}^{t} \nabla^2 A^0(\vec{x}, t')\,dt' = \vec{\nabla} \cdot A'(\vec{x}, t) \equiv F(\vec{x}, t) \neq 0$$

Now make a second gauge transformation

$$A_\mu'' = A_\mu' + \partial_\mu \Lambda', \quad \Lambda'(\vec{x}, t) = \int d^3x' \, \frac{F(\vec{x}', t)}{4\pi |\vec{x} - \vec{x}'|}$$

$$A''_0 = \partial_0 \lambda' = \int d^3x' \, \frac{\partial}{\partial t} F(\vec{x}', t) \, \frac{1}{4\pi |\vec{x} - \vec{x}'|}$$

But $\frac{\partial}{\partial t} F(\vec{x}, t) = \vec{\nabla} \cdot \dot{\vec{A}} + \nabla^2 A^0 (\vec{x}, t) = -\vec{\nabla} \cdot \vec{E}(\vec{x}, t)$

$= -\rho(\vec{x}, t)$

$$A''_0 = -\int d^3x \, \frac{\rho(\vec{x}', t)}{4\pi |\vec{x} - \vec{x}'|} \quad \text{or} \quad A''^0 = \boxed{\varphi = \int d^3x \, \frac{\rho(\vec{x}', t)}{4\pi |\vec{x} - \vec{x}'|}}$$

Coulomb potential

$$\vec{A}'' = \vec{A}' + \vec{\nabla} \lambda'$$

$$\vec{\nabla} \cdot \vec{A}'' = \vec{F}(x, t) + \nabla^2 \int d^3x' \, \frac{F(\vec{x}', t)}{4\pi |\vec{x} - \vec{x}'|}$$

Now recall

$$\nabla^2 \frac{1}{4\pi |\vec{x} - \vec{x}'|} = -\delta(\vec{x} - \vec{x}')$$

$$\boxed{\vec{\nabla} \cdot \vec{A}'' = 0.}$$

This is the Coulomb or radiation gauge.

($A^0 = 0$ only if $\rho = 0$)

3-3

When $\vec{\nabla}\cdot\vec{H} = g\,\delta(\vec{r})$, $\vec{H}$ cannot be the curl of a vector potential, but we can write

$$\vec{H} = \vec{\nabla}\times\vec{A} + g\,\vec{f}(\vec{r})$$

where $\vec{f}$ is a ("string") function satisfying

$$\vec{\nabla}\cdot\vec{f} = \delta(\vec{r}),$$

while $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ as usual. Consider now

$$\int (d\vec{r}\,')\,\vec{f}(\vec{r}-\vec{r}\,')\times\vec{H}(\vec{r}\,')$$

$$= \int (d\vec{r}\,')\Big[\vec{f}(\vec{r}-\vec{r}\,')\times(\vec{\nabla}'\times\vec{A}(\vec{r}\,'))$$

$$- g\,\vec{f}(\vec{r}-\vec{r}\,')\times\vec{f}(\vec{r}\,')\Big]$$

The last term is zero:

$$\int (d\vec{r}\,')\,\vec{f}(\vec{r}-\vec{r}\,')\times\vec{f}(\vec{r}\,') =$$

$$= \int (d\vec{r}\,')\,\vec{f}(\vec{r}\,')\times\vec{f}(\vec{r}-\vec{r}\,') \qquad \left\{\begin{array}{l}\vec{r}-\vec{r}\,'\to\vec{r}\,'\\ \vec{r}\,'\to\vec{r}-\vec{r}\,'\end{array}\right.$$

$$= -\int (d\vec{r}\,')\,\vec{f}(\vec{r}-\vec{r}\,')\times\vec{f}(\vec{r}\,')$$

$$\therefore \;= 0$$

For the other term we integrate by parts:

$$\vec{f}(\vec{r}-\vec{r}') \times (\vec{\nabla}' \times \vec{A}(\vec{r}'))$$

$$= \vec{f}\cdot(\vec{\nabla}')\cdot\vec{A} - (\vec{f}\cdot\vec{\nabla}')\vec{A}$$

$$= \vec{\nabla}'[\vec{f}\cdot\vec{A}] - (\vec{\nabla}'\vec{f})\cdot\vec{A} - \vec{\nabla}'\cdot[\vec{f}\vec{A}] + (\vec{\nabla}'\cdot\vec{f})\vec{A}$$

$\int(d\vec{r}')$:

$$\int(d\vec{r}')\vec{\nabla}'[\vec{f}\cdot\vec{A}] = \int d\vec{\sigma}'\vec{f}(\vec{r}-\vec{r}')\cdot\vec{A}(\vec{r}') = 0$$

$$\int(d\vec{r}')\vec{\nabla}'\cdot[\vec{f}\vec{A}] = \int d\vec{\sigma}'\cdot\vec{f}(\vec{r}-\vec{r}')\vec{A}(\vec{r}') = 0$$

since, as we'll see, $\vec{f}$ is zero except on a line, intersecting the surface at one (or more) points, a set of measure zero.

We are left with

$$-\int(d\vec{r}')\underbrace{(\vec{\nabla}'\vec{f}(\vec{r}-\vec{r}'))}_{-\vec{\nabla}\vec{f}(\vec{r}-\vec{r}')}\cdot\vec{A}(\vec{r}')$$

$$= \vec{\nabla}\int(d\vec{r}')\vec{f}(\vec{r}-\vec{r}')\cdot\vec{A}(\vec{r}')$$

and $$\int(d\vec{r}')\underbrace{\vec{\nabla}'\cdot\vec{f}(\vec{r}-\vec{r}')}_{\substack{-\vec{\nabla}\cdot\vec{f}(\vec{r}-\vec{r}') \\ = -\delta(\vec{r}-\vec{r}')}}\vec{A}(\vec{r}')$$

$$= \vec{A}(\vec{r})$$

So: $\vec{A}$ lr. $= \vec{\nabla}\int(d\vec{r}')\,\vec{f}(\vec{r}-\vec{r}')\cdot\vec{A}(\vec{r}')$

$$- \int(d\vec{r}')\,\vec{f}(\vec{r}-\vec{r}')\times\vec{H}(\vec{r}')$$

$$\equiv - \int(d\vec{r}')\,\vec{f}(\vec{r}-\vec{r}')\times\vec{H}(\vec{r}') + \vec{\nabla}\lambda$$

where the last form makes the analogy with a gauge potential. We can carry this a bit further by supposing, perhaps not totally naturally, that $\vec{A}(\vec{r})$ has time dependence as well. Then, in order that $\vec{E}(\vec{r}) = 0$ we require

$$\vec{E} = -\vec{\nabla}\varphi - \dot{\vec{A}} = -\nabla\varphi - \frac{\partial}{\partial t}\vec{\nabla}\lambda(\vec{r},t) = 0$$

whence $\quad \varphi = -\frac{\partial}{\partial t}\lambda(\vec{r},t)$

changing the form of $\vec{f}$ (reorienting the string) is a kind of gauge transformation. [This last becomes much more transparent when motion of the magnetic charge is allowed, so $\vec{E} \neq 0$.]

Now we need an explicit form for $\vec{f}$.
Compare $\vec{\nabla}\cdot\vec{f}(\vec{r}) = \delta(\vec{r})$ with $\vec{\nabla}\cdot\vec{j}(\vec{r}) = \dot{\rho}(\vec{r})$
The first eqn thus says that "charge" is being created at constant rate at the origin. This can only be true if "current" flows in from infinity. It is familiar from classical electrodynamics that such a current is given by the line integral $\vec{f}(\vec{r}) = \int_C d\vec{r}'\ \delta(\vec{r}-\vec{r}')$, C is a path from 0 to ∞.
Verification: $\vec{\nabla}\cdot\vec{f}(\vec{r}) = -\int_C d\vec{r}'\cdot\vec{\nabla}'\,\delta(\vec{r}-\vec{r}')$

$$= -\ \delta(\vec{r}-\vec{r}')\Big|_{\vec{r}'=0}^{\vec{r}=\infty} = \delta(\vec{r})\ \checkmark$$

The simplest choice for C is a straight line with orientation $\hat{n}$: $d\vec{r}' = \hat{n}\,dl$ where dl is the element of length along the line.

Let's then compute $\vec{A}(\vec{r})$ in the gauge $\lambda = 0$ (we'll check consistency below).

$$\vec{A}(\vec{r}) = -\int (d\vec{r}')\,\vec{f}(\vec{r}-\vec{r}')\times\vec{H}(\vec{r}')$$

where f[illegible] point magnetic charge at the origin 9.

$$\vec{H} = \frac{g}{4\pi}\frac{\vec{r}}{r^3}$$

With the above choice for $\vec{f}$,

$$\vec{A}(\vec{r}) = -\int(d\vec{r}\,')\int_0^\infty dl\;\delta(\vec{r}-\vec{r}\,'-\hat{n}l)\,\frac{g}{4\pi}\,\frac{\hat{n}\times\vec{r}\,'}{r'^3}$$

$$= -\int_0^\infty dl\;\frac{g}{4\pi}\,\frac{\vec{n}\times\vec{r}}{|\vec{r}-\hat{n}l|^3}$$

Now evaluate $\displaystyle\int_0^\infty dl\,\frac{1}{|\vec{r}-\hat{n}l|^3} = \int_0^\infty dl\,\frac{1}{(r^2+l^2-2l\hat{n}\cdot\vec{r})^{3/2}}$

$$= \int_0^\infty dl\,\frac{1}{[(l-\hat{n}\cdot\vec{r})^2+r^2-(\hat{n}\cdot\vec{r})^2]^{3/2}} \quad = I$$

where we see $\displaystyle\int_b^\infty dz\,\frac{1}{(z^2+a^2)^{3/2}}$, an elementary integral.

Alternatively we note that

$$\left(r\frac{\partial}{\partial\hat{n}\cdot\vec{r}} + \hat{n}\cdot\vec{r}\,\frac{\partial}{\partial r}\right) I(\hat{n}\cdot\vec{r}, r) = \frac{1}{r^2} \quad (*)$$

which has solution $\displaystyle I = \frac{1}{r(r-\hat{n}\cdot\vec{r})}$

which has a singularity along the semi-infinite line given by $\hat{n}$. (The homog. sln to * has sing. along infinite line)

Thus

$$\vec{A}(\vec{r}) = -\frac{g}{4\pi}\frac{1}{r}\frac{\hat{n}\times\vec{r}}{r-\hat{n}\cdot\vec{r}}$$

and, if $\hat{n}=\hat{z}$,

$$\vec{A} = -\frac{g}{4\pi r}\frac{\hat{z}\times\vec{r}}{r-r\cos\theta}$$

$$\hat{z}\times\vec{r} = r\hat{\varphi}\sin\theta$$

$$\vec{A} = -\frac{g}{4\pi r}\hat{\varphi}\frac{\sin\theta}{1-\cos\theta} = -\frac{g}{4\pi r}\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\underset{\sin}{\cancel{\cos}}^2\frac{\theta}{2}}\hat{\varphi}$$

$$= -\frac{g}{4\pi r}\cot\frac{\theta}{2}\,\hat{\varphi}$$

$\lambda = 0$ since $\vec{f}\cdot\vec{A}=0$