Homework II

5. a) Assuming that P_ν & $J_{\lambda\sigma}$ are vector and tensor quantities respectively,

$$P_\nu \to P_\nu - \delta\omega_\nu{}^\lambda P_\lambda$$

$$J_{\lambda\sigma} \to J_{\lambda\sigma} - \delta\omega_\lambda{}^\rho J_{\rho\sigma} - \delta\omega_\sigma{}^\rho J_{\lambda\rho},$$

it is easy to prove that

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} J_{\lambda\sigma} P_\nu$$

is a vector:

$$W^\mu \to \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\left[J_{\lambda\sigma} - \delta\omega_\lambda{}^\rho J_{\rho\sigma} - \delta\omega_\sigma{}^\rho J_{\lambda\rho}\right] \times \left[P_\nu - \delta\omega_\nu{}^\tau P_\tau\right]$$

$$= W^\mu - \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\left[\delta\omega_\lambda{}^\rho J_{\rho\sigma} P_\nu + \delta\omega_\sigma{}^\rho J_{\lambda\rho} P_\nu + \delta\omega_\nu{}^\tau J_{\lambda\sigma} P_\tau\right]$$

$$= W^\mu - \frac{1}{2} J_{\rho\sigma} P_\nu \delta\omega_{\lambda\tau} \left[\varepsilon^{\mu\nu\lambda\sigma} g^{\tau\rho} + \varepsilon^{\mu\nu\rho\lambda} g^{\tau\sigma} + \varepsilon^{\mu\lambda\rho\sigma} g^{\tau\nu} \right]$$

Now we use the identity

$$\varepsilon^{\mu\nu\lambda\sigma} g^{\tau\rho} + \varepsilon^{\nu\lambda\sigma\rho} g^{\tau\mu} + \varepsilon^{\lambda\sigma\rho\mu} g^{\tau\nu} + \varepsilon^{\sigma\rho\mu\nu} g^{\tau\lambda} + \varepsilon^{\rho\mu\nu\lambda} g^{\tau\sigma} = 0$$

(prove just like the lemma in prob. 4)

$$\text{So } W^\mu \to W^\mu + \frac{1}{2} J_{\rho\sigma} P_\nu \delta\omega_{\lambda\tau} \left[\varepsilon^{\nu\lambda\sigma\rho} g^{\tau\mu} + \varepsilon^{\sigma\rho\mu\nu} \cancel{g^{\tau\lambda}} \right]$$

$$= W^\mu + \frac{1}{2} \varepsilon^{\lambda\nu\rho\sigma} J_{\rho\sigma} P_\nu \delta\omega_\lambda{}^\mu$$

$$= W^\mu - \delta\omega^\mu{}_\lambda W^\lambda \quad \checkmark$$

b) For a massless particle $P^0 = |\vec{P}|$ and

$$W^0 = \frac{1}{2} \varepsilon^{0ijk} J_{jk} P_i = J_i P_i = \vec{S} \cdot \vec{P}$$

$$= \lambda P^0 \text{ provided}$$

$$\lambda = \frac{\vec{S} \cdot \vec{P}}{P^0}$$

$$W^i = \frac{1}{2}\varepsilon^{i0jk} J_{jk} P_0 + \varepsilon^{ij0k} J_{0k} P_j$$

$$= - J_i P_0 - \varepsilon^{ijk} N_k P_j$$

or $\vec{W} = -P_0 \vec{J} - \vec{P} \times \vec{N} = |\vec{P}|\vec{S} - \vec{P} \times \vec{N}$

But $(\vec{S} \times \vec{P}) \times \vec{P} = \vec{P}(\vec{S} \cdot \vec{P}) - (\vec{P})^2 \vec{S}$

$\therefore\ |\vec{P}|\vec{W} = -(\vec{S} \times \vec{P}) \times \vec{P} + \vec{P}(\vec{S} \cdot \vec{P}) - \vec{P} \times \vec{N}\,|\vec{P}|$

$$\vec{W} = \lambda \vec{P} + \vec{P} \times \left[\frac{\vec{S} \times \vec{P}}{|\vec{P}|} - \vec{N} \right]$$

Since $W^2 = \vec{W}^2 - W^{02} = 0$

$$= \lambda^2 \underset{0}{\underset{\shortparallel}{P^2}} + \left[\vec{P} \times \left(\frac{\vec{S} \times \vec{P}}{|\vec{P}|} - \vec{N} \right) \right]^2$$

for $m \to 0$, $\quad \vec{N} = \dfrac{\vec{S} \times \vec{P}}{|\vec{P}|} + (\quad)\vec{P}$

$$\vec{W} = \lambda \vec{P}, \quad W^\mu = \lambda P^\mu$$

2. $$u_{p\sigma} = \frac{1}{\sqrt{2m(E+m)}}(m - \gamma p)v_\sigma$$

$$u^\dagger_{p\sigma}\gamma^0 = v^\dagger_\sigma \gamma^0 (m - \gamma p)\frac{1}{\sqrt{2m(E+m)}}$$

since $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu$

$$\sum_\sigma u_{p\sigma} u^\dagger_{p\sigma}\gamma^0 = \frac{1}{2m(E+m)}(m-\gamma p)\underbrace{\sum_\sigma v_\sigma v^\dagger_\sigma \gamma^0}_{\frac{1}{2}(1+\gamma^0)}(m-\gamma p)$$

Now $(1+\gamma^0)(m-\gamma p) = (m-\gamma p) + (m+\gamma p + 2\gamma^0 E)\gamma^0$

$$(m-\gamma p)^2 = m^2 \underbrace{- p^2}_{+m^2} - 2m\gamma p = 2m(m-\gamma p)$$

$$(m-\gamma p)(m+\gamma p) = m^2 + p^2 = 0$$

So: $$\sum_\sigma u_{p\sigma} u^\dagger_{p\sigma}\gamma^0 = \frac{1}{4m}\frac{1}{E+m}(m-\gamma p)\left[2m + 2E\right]$$

$$= \frac{m-\gamma p}{2m} \checkmark$$

The normalization condition is

$$u^\dagger_{p\sigma}\gamma^0\gamma^\mu u_{p\sigma'} = \frac{1}{2m(E+m)} v^\dagger_\sigma \gamma^0 (m-\gamma p)\gamma^\mu(m-\gamma p)v_{\sigma'}$$

Now $(m-\gamma p)\gamma^\mu = \gamma^\mu(m+\gamma p) + 2p^\mu$

$(m+\gamma p)(m-\gamma p) = 0$, so since

$$v_\sigma^+ \gamma^0 \underbrace{(m-\gamma p)}_{m+\gamma^0 E - \vec{\gamma}\cdot\vec{p}} v_{\sigma'} = (m+E)\delta_{\sigma\sigma'}$$

because $\gamma^0 v_{\sigma'} = v_{\sigma'}$, $v_\sigma^+\gamma^0 = v_\sigma^+$

and $v_\sigma^+ \vec{\gamma}\, v_{\sigma'} = v_\sigma^+ \vec{\gamma}\gamma^0 v_{\sigma'} = -v_\sigma^+ \gamma^0\vec{\gamma}\, v_{\sigma'}$

$= -v_\sigma^+ \vec{\gamma}\, v_{\sigma'} = 0$

Therefore

$$u_{p\sigma}^+ \gamma^0\gamma^\mu u_{p\sigma'} = \frac{p^\mu}{m}\delta_{\sigma\sigma'} \quad \checkmark$$

3. Define $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$

$$\gamma_5^{\,2} = \underbrace{\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0}_{3\text{ antiom's}}\gamma^1\gamma^2\gamma^3 = -\underbrace{\gamma^1\gamma^2\gamma^3\gamma^1}_{2\text{ antiom's}}\gamma^2\gamma^3$$

$\boxed{\gamma^{0\,2} = -\gamma^{1\,2} = -\gamma^{2\,2} = -\gamma^{3\,2} = 1}$

$$= +\gamma^2\gamma^3\gamma^2\gamma^3 = +\gamma^3\gamma^3 = -1, \text{ so}$$

$$(i\gamma_5)^2 = +1 \quad \checkmark$$

$\{\gamma_5, \gamma_\mu\} = 0$ because, $\mu = 0, 1, 2,$ or 3, γ_μ anticommutes with 3 of the γ's and commutes with the fourth. Finally

$$\gamma_5^{+} = \gamma^{3+}\gamma^{2+}\gamma^{1+}\gamma^{0+} = (-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0$$

$$= -(-\gamma^0\gamma^3\gamma^2\gamma^1) = +\gamma^0\gamma^1\gamma^3\gamma^2$$

$$= -\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma_5$$

so $(i\gamma_5)^{+} = i\gamma_5$

4. Let the rest frame spinors be defined by

$$\gamma^0 v_\sigma = v_\sigma$$

$$\Sigma_3 v_\sigma = \sigma v_\sigma\,, \quad \sigma = \pm 1$$

$$\tfrac{1}{2}(\Sigma_1 \pm i\Sigma_2)\, v_\mp = v_\pm$$

Then if γ^0 and $\vec{\Sigma}$ are imaginary, as they are in our representation, then

$$\gamma^0 v_\sigma^* = -v_\sigma^*$$

$$\Sigma_3 v_\sigma^* = -\sigma v_\sigma^*$$

$$\tfrac{1}{2}(\Sigma_1 \mp i\Sigma_2)\, v_\mp^* = -v_\pm^*$$

These relations are satisfied if $v_\sigma^* = -i\gamma_5\sigma v_{-\sigma}$

$$\gamma^0(-i\gamma_5\sigma v_{-\sigma}) = +i\gamma_5\sigma\gamma^0 v_{-\sigma} = +i\gamma_5\sigma v_{-\sigma} \checkmark$$

$$\Sigma_3(-i\gamma_5\sigma v_{-\sigma}) = -i\gamma_5\sigma\Sigma_3 v_{-\sigma} \quad \text{since } [\gamma_5,\vec{\Sigma}]=0$$

$$= -i\gamma_5\sigma(-\sigma)v_{-\sigma}$$

$$= -\sigma(-i\gamma_5\sigma v_{-\sigma}) \checkmark$$

$$\frac{1}{2}(\Sigma_1 \mp i\Sigma_2)(-i\gamma_5(\mp)v_{\pm})$$

$$= \pm i\gamma_5 \underbrace{(\Sigma_1 \mp i\Sigma_2)v_{\pm}}_{v_{\mp}} = -(-i\gamma_5(\pm)v_{\mp}) \checkmark$$

5. Under a Lorentz transformation

$$\psi \to \left(1 - \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\psi$$

$$\bar{\psi} \to \psi^{+}\left(1 + \frac{i}{4}\sigma^{\alpha\beta +}\delta\omega_{\alpha\beta}\right)\gamma^{0}$$

$$= \bar{\psi}\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)$$

since $\sigma^{\alpha\beta +}\gamma^{0} = \gamma^{0}\sigma^{\alpha\beta}$

Thus $\bar{\psi}\psi \to \bar{\psi}\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\left(1 - \frac{i}{4}\sigma^{\gamma\delta}\delta\omega_{\gamma\delta}\right)\psi$

$= \bar{\psi}\psi$ to first order in $\delta\omega_{\alpha\beta}$

<u>$\bar{\psi}\psi$ is therefore a scalar.</u>

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \bar{\psi}\left(1 + \frac{i}{4}\sigma^{\alpha\beta}\delta\omega_{\alpha\beta}\right)\sigma^{\mu\nu} \otimes \left(1 - \frac{i}{4}\sigma^{\gamma\delta}\delta\omega_{\gamma\delta}\right)\psi$$

$$= \bar{\psi}\left[\sigma^{\mu\nu} + \frac{i}{4}\cdot\left[\sigma^{\alpha\beta}, \sigma^{\mu\nu}\right]\delta\omega_{\alpha\beta}\right]\psi$$

Now $J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ satisfies [Recall Hwk I]

$$[J_{\mu\nu}, J_{\kappa\lambda}] = i\left[g_{\mu\kappa}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\kappa} - g_{\nu\kappa}J_{\mu\lambda} + g_{\nu\lambda}J_{\mu\kappa}\right]$$

$$
\begin{aligned}
\therefore\ \delta\, \bar{\Psi}\sigma^{\mu\nu}\psi = &-\delta\omega^{\mu\beta}\,\bar{\Psi}\tfrac{1}{2}\sigma_{\beta}{}^{\nu}\psi \\
&+\delta\omega^{\nu\beta}\,\bar{\Psi}\tfrac{1}{2}\sigma_{\beta}{}^{\mu}\psi \\
&+\delta\omega^{\alpha\mu}\,\bar{\Psi}\tfrac{1}{2}\sigma_{\alpha}{}^{\nu}\psi \\
&-\delta\omega^{\alpha\nu}\,\bar{\Psi}\tfrac{1}{2}\sigma_{\alpha}{}^{\mu}\psi \\
= &-\delta\omega^{\mu\beta}\,\bar{\Psi}\sigma_{\beta}{}^{\nu}\psi \\
&-\delta\omega^{\nu\beta}\,\bar{\Psi}\sigma^{\mu}{}_{\beta}\psi
\end{aligned}
$$

which is just like a second-rank tensor transforms (transforms as a vector in each index).