Physics 5970
Homework Assignment I

1. Let $\beta = \tau_3 \times \sigma_2$, $\alpha_1 = 1 \times \sigma_1$, $\alpha_2 = \tau_2 \times \sigma_2$, $\alpha_3 = 1 \times \sigma_3$

$$\beta^2 = \tau_3^2 \times \sigma_2^2 = 1 \times 1 = 1 \text{, likewise } \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$$

$$\{\beta, \alpha_1\} = (\tau_3 \times \sigma_2)(1 \times \sigma_1) + (1 \times \sigma_1)(\tau_3 \times \sigma_2)$$

$$= \tau_3 \times (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) = 0$$

$$\{\beta, \alpha_2\} = (\tau_3 \times \sigma_2)(\tau_2 \times \sigma_2) + (\tau_2 \times \sigma_2)(\tau_3 \times \sigma_2)$$

$$= (\tau_3 \tau_2 + \tau_2 \tau_3) \times \sigma_2^2 = 0$$

$$\{\beta, \alpha_3\} = (\tau_3 \times \sigma_2)(1 \times \sigma_3) + (1 \times \sigma_3)(\tau_3 \times \sigma_2)$$

$$= \tau_3 \times (\sigma_2 \sigma_3 + \sigma_3 \sigma_2) = 0$$

from properties of Pauli matrices

Similarly $\{\alpha_1, \alpha_2\} = (1 \times \sigma_1)(\tau_2 \times \sigma_2) + (\tau_2 \times \sigma_2)(1 \times \sigma_1)$

$$= \tau_2 \times (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) = 0$$

$$\{\alpha_1, \alpha_3\} = (1 \times \sigma_1)(1 \times \sigma_3) + (1 \times \sigma_3)(1 \times \sigma_1)$$

$$= 1 \times (\sigma_1 \sigma_3 + \sigma_3 \sigma_1) = 0$$

$$\{\alpha_2, \alpha_3\} = (\tau_2 \times \sigma_2)(1 \times \sigma_3) + (1 \times \sigma_3)(\tau_2 \times \sigma_2)$$

$$= \tau_2 \times (\sigma_2 \sigma_3 + \sigma_3 \sigma_2) = 0$$

This verifies that the Dirac algebra is satisfied. Obvious $\beta, \vec{\alpha}$ are Hermitian, being the direct product of Hermitian matrices. Finally

$$\beta^* = \tau_3^* \times \sigma_2^* = \tau_3 \times (-\sigma_2) = -\beta \checkmark$$

$$\alpha_1^* = 1^* \times \sigma_1^* = 1 \times \sigma_1 = \alpha_1$$

$$\alpha_2^* = \tau_2^* \times \sigma_2^* = (-\tau_2) \times (-\sigma_2) = \alpha_2$$

$$\alpha_3^* = 1^* \times \sigma_3^* = 1 \times \sigma_3 = \alpha_3 \qquad \checkmark$$

2. Let $\frac{1}{2}[\alpha_i, \alpha_j] = i \varepsilon_{ijk} \Sigma_k$

or since $\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{jk} = 2\delta_{kl}$

$$\Sigma_k = \frac{1}{4i} \varepsilon_{ijk} [\alpha_i, \alpha_j]$$
$$= \frac{1}{2i} \varepsilon_{ijk} \alpha_i \alpha_j \quad \text{since } \varepsilon_{ijk} = -\varepsilon_{jik}$$

Then $\Sigma_k \Sigma_l = -\frac{1}{4} \varepsilon_{ijk} \varepsilon_{rsl} \alpha_i \alpha_j \alpha_r \alpha_s$

There are only three α's so at least two of these must be the same, say $i=r$ (equivalently $j=r$, $i=s$, $j=s$; other possibilities give 0)

$$\Sigma_k \Sigma_l = -\frac{1}{4} \underbrace{\varepsilon_{ijk}\,\varepsilon_{isl}}_{\delta_{js}\delta_{kl}-\delta_{jl}\delta_{ks}.} \underbrace{\alpha_i\,\alpha_j\,\alpha_i}_{-\alpha_i^2\alpha_j}\,\alpha_s \times \begin{cases} 4 \text{ if } j\neq s \\ 2 \text{ if } j=s \end{cases} \quad \text{since } i \neq j$$

$$= \left(\tfrac{2}{3}\delta_{js}\delta_{kl} - \delta_{jl}\delta_{ks}\right)\alpha_j\alpha_s$$

$$= \tfrac{2}{3}\delta_{kl}\underbrace{\vec{\alpha}\cdot\vec{\alpha}}_{3} - \alpha_l\alpha_k$$

$$\alpha_l\alpha_k = \tfrac{1}{2}\{\alpha_l,\alpha_k\} + \tfrac{1}{2}[\alpha_l,\alpha_k]$$

$$= \delta_{kl} + i\,\varepsilon_{lkm}\,\Sigma_m$$

$$\therefore\ \Sigma_k\Sigma_l = \delta_{kl} + i\,\varepsilon_{klm}\,\Sigma_m \quad \checkmark$$

Explicitly: $\Sigma_1 = \frac{1}{2i}\varepsilon_{1jk}\,\alpha_j\alpha_k = \frac{1}{i}\alpha_2\alpha_3$

$$= \tfrac{1}{i}(\tau_1\times\sigma_2)(1\times\sigma_3) = \tfrac{1}{i}\tau_2\times i\sigma_1$$

$$= \tau_2\times\sigma_1$$

$$\Sigma_2 = \tfrac{1}{2i}\varepsilon_{2jk}\,\alpha_j\alpha_k = \tfrac{1}{i}\alpha_3\alpha_1$$

$$= \tfrac{1}{i}(1\times\sigma_3)(1\times\sigma_1) = \tfrac{1}{i}(1\times\sigma_3\sigma_1) = 1\times\sigma_2$$

and $\Sigma_3 = \frac{1}{i}\alpha_1\alpha_2 = \frac{1}{i}(1\times\sigma_1)(\tau_2\times\sigma_2)$

$$= \tau_2 \times \sigma_3$$

Check $\Sigma_1\Sigma_2 = (\tau_2\times\sigma_1)(1\times\sigma_2)$

$$= \tau_2\times\sigma_1\sigma_2 = i\,\tau_2\times\sigma_3 = i\Sigma_3$$

3. Verify rotational invariance of the Dirac equation: If in the primed frame,

$$i\frac{\partial}{\partial t'}\bar{\Psi}(\vec{x}',t') = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla}' + \beta m\right)\bar{\Psi}(\vec{x}',t')$$

where $t' = t$, $\vec{x}' = \vec{x} - \delta\vec{\omega}\times\vec{x}$

$$\vec{\nabla}' = \vec{\nabla} - \delta\vec{\omega}\times\vec{\nabla}$$

$$\bar{\Psi}(\vec{x}',t) = U\,\Psi(\vec{x},t)$$

$$= \left(1 + i\,\delta\vec{\omega}\cdot\tfrac{1}{2}\vec{\Sigma}\right)\Psi$$

then $\left(1 + i\,\delta\vec{\omega}\cdot\frac{1}{2}\vec{\Sigma}\right) i\frac{\partial}{\partial t}\Psi(\vec{x},t)$

$$= \left(1 + i\,\delta\vec{\omega}\cdot\tfrac{1}{2}\vec{\Sigma}\right)\left(\vec{\alpha}\tfrac{1}{i}\cdot\vec{\nabla} + \beta m\right)\Psi(\vec{x},t)$$

$$- \tfrac{1}{i}\vec{\alpha}\cdot\delta\vec{\omega}\times\vec{\nabla}\,\Psi + \left[\tfrac{1}{i}\vec{\alpha}, +i\,\delta\vec{\omega}\cdot\tfrac{1}{2}\vec{\Sigma}\right] \otimes \vec{\nabla}\Psi$$

But $[\vec{\alpha}, \delta\vec{\omega}\cdot\vec{\Sigma}]_i$

$$= \left[\alpha_i , \frac{1}{2i}\delta\omega_j \varepsilon_{jkl}\alpha_k\alpha_l\right]$$

$$= \frac{1}{2i}\delta\omega_j \varepsilon_{jkl}\left(\underbrace{\{\alpha_i,\alpha_k\}}_{2\delta_{ik}}\alpha_l - \alpha_k\underbrace{\{\alpha_i,\alpha_l\}}_{2\delta_{il}}\right)$$

$$= \frac{2}{i}\delta\omega_j \varepsilon_{jil}\alpha_l = 2i(\delta\vec{\omega}\times\vec{\alpha})_i$$

The last two terms on p. 4 are

$$i\alpha\cdot\delta\vec{\omega}\times\vec{\nabla}\psi + i\,\delta\vec{\omega}\times\vec{\alpha}\cdot\vec{\nabla}\psi = 0$$

since $\vec{\alpha}\cdot\delta\vec{\omega}\times\vec{\nabla} = \vec{\alpha}\times\delta\vec{\omega}\cdot\vec{\nabla}$

4. Since $G = \frac{1}{2}J_{\mu\nu}\delta\omega^{\mu\nu} = \delta\vec{\omega}\cdot\vec{J} + \delta\vec{v}\cdot\vec{N}$

$$\delta\omega^{ij} = \varepsilon^{ijk}\delta\omega_k , \quad \delta\omega^{0i} = \delta v^i$$

$$\boxed{\begin{aligned} &\delta x^\mu = \delta\omega^{\nu\mu}x_\nu \\ &\Rightarrow \delta\vec{x} = \delta\vec{\omega}\times\vec{x} \quad R \\ &\left.\begin{aligned}\delta t &= -\delta\vec{v}\cdot\vec{x} \\ \delta\vec{x} &= -\delta\vec{v}\,t\end{aligned}\right\} B \end{aligned}}$$

$$\frac{1}{2}J_{ij}\varepsilon^{ijk} = J_k , \quad J_{ij} = \varepsilon_{ijk}J_k$$

$$J_{0i} = -J^0{}_i = N_i$$

Then, since $[J_i, J_j] = i\varepsilon_{ijk}J_k$

$$[J_i, N_j] = i\varepsilon_{ijk}N_k , \quad [N_i, N_j] = -i\varepsilon_{ijk}J_k$$

consider $[J_{12}, J_{13}] = [J_3, -J_2]$

$$= iJ_1 = iJ_{23}$$

$$[J_{12}, J_{30}] = [J_3, -N_3] = 0$$

$$[J_{12}, J_{10}] = [J_3, -N_1] = -iN_2 = -iJ_{02}$$

$$[J_{10}, J_{20}] = [-N_1, -N_2] = -iJ_3 = -iJ_{12}$$

By cyclic permutation we can deduce all the others. Since exactly two indices must be equal (otherwise we get zero), we therefore have

$$[J_{\mu\nu}, J_{\kappa\lambda}] = i[g_{\mu\kappa} J_{\nu\lambda} - g_{\nu\kappa} J_{\mu\lambda} - g_{\mu\lambda} J_{\nu\kappa} + g_{\nu\lambda} J_{\mu\kappa}]$$

since it must be antisymmetric under $\mu \leftrightarrow \nu$, $\kappa \leftrightarrow \lambda$.

<u>Alternative</u> argument

$$[J_{ij}, J_{kl}] = \varepsilon_{ijm} \varepsilon_{kln} [J_m, J_n]$$

$$= \varepsilon_{ijm} \varepsilon_{kln}\, i \varepsilon_{mnp} J_p$$

Lemma: $\varepsilon_{ijm} \delta_{nq} - \varepsilon_{jmn} \delta_{iq} + \varepsilon_{mni} \delta_{jq} - \varepsilon_{nij} \delta_{mq} = 0$

Proof: evidently the LHS $=0$ if $i=j$, $i=m$, or $j=m$. If $i\neq j\neq m$ say $i=1, j=2, m=3$ LHS $=$

$$\varepsilon_{123}\delta_{nq}-\varepsilon_{23n}\delta_{1q}+\varepsilon_{3n1}\delta_{2q}-\varepsilon_{n12}\delta_{3q}$$

$$=\delta_{nq}-\delta_{1n}\delta_{1q}-\delta_{2n}\delta_{2q}-\delta_{3n}\delta_{3q}=0 \checkmark$$

Then
$$[J_{ij}, J_{kl}] = i[\varepsilon_{jmn}\delta_{iq}-\varepsilon_{mni}\delta_{jq}+\varepsilon_{nij}\delta_{mq}]\varepsilon_{mqp}\varepsilon_{kln}J_p$$

$$= i[\varepsilon_{jmn}\varepsilon_{kln}J_{mi}-\varepsilon_{mni}\varepsilon_{kln}J_{mj}+\varepsilon_{nij}\varepsilon_{kln}\cancel{J_{mm}}\xrightarrow{}0]$$

$$= i[(\delta_{jk}\delta_{ml}-\delta_{jl}\delta_{mk})J_{mi}-(\delta_{ml}\delta_{ik}-\delta_{mk}\delta_{il})J_{mj}]$$

$$= i[\delta_{jk}J_{li}-\delta_{jl}J_{ki}-\delta_{ik}J_{lj}+\delta_{il}J_{kj}]$$

$$= i[\delta_{ik}J_{jl}-\delta_{il}J_{jk}-\delta_{jk}J_{il}+\delta_{jl}J_{ik}$$

which is the space version of desired result.

$$[J_{ij}, J_{k0}] = \varepsilon_{ijl}[J_l, -N_k]$$

$$= -\varepsilon_{ijl}\, i\, \varepsilon_{lkp} N_p$$

$$= -i(\delta_{ik}\delta_{jp} - \delta_{ip}\delta_{jk}) J_{0p}$$

$$= -i(\delta_{ik} J_{0j} - \delta_{jk} J_{0i})$$

$$= i[\delta_{ik} J_{j0} - \delta_{jk} J_{i0}]$$

which is the desired time-space result. Finally

$$[J_{i0}, J_{j0}] = [-N_i, -N_j] = -i\varepsilon_{ijk} J_k$$

$$= -iJ_{ij} = i g_{00} J_{ij}$$

which completes the proof.

For spin-$\frac{1}{2}$: $\vec{J} = \frac{1}{2}\vec{\Sigma}$, $\vec{\Sigma} = \frac{1}{2i}\vec{\alpha}\times\vec{\alpha}$

$$\vec{N} = +\frac{i}{2}\vec{\alpha}$$

so $J_{ij} = \varepsilon_{ijk}\frac{1}{2}\Sigma_k = \frac{-i}{4}[\alpha_i, \alpha_j] = \frac{i}{4}[\gamma_i, \gamma_j]$

since $\gamma^0\alpha_i\gamma^0\alpha_j - \gamma^0\alpha_j\gamma^0\alpha_i = -\alpha_i\alpha_j + \alpha_j\alpha_i$

and $J^0{}_i = -N_i = \frac{i}{2}\alpha_i = \frac{i}{4}[\gamma^0, \gamma_i]$

since $\frac{1}{2}[\gamma^0, \gamma^0\alpha_i] = \frac{1}{2}\gamma^{0\,2}\alpha_i - \frac{1}{2}\gamma^0\alpha_i\gamma^0 = \alpha_i$

$\therefore$ $J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$ where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

and $\sigma^{ij} = \varepsilon_{ijk}\Sigma_k$, $\frac{1}{2}\sigma^{0i} = \frac{i}{2}\alpha_i = -N_i$
