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Matrix models with $\gamma_{\text{string}} > 0$

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Within the framework of $c=1$ matrix models, we consider multi-matrix models, i.e. the quantum mechanics of multi-matrix models. A connection is established between a D-matrix model and a D-dimensional gas of fermions (bosons) for odd (even) values of D. A statistical mechanical analysis yields the scaling law for the free energy, and hence the susceptibility exponents for the various models. The exponents turn out to be positive for the multi-matrix models, suggesting that these could represent models of 2D gravity coupled to $c>1$ matter. However, a lower-bound on the mass-gap exponents is found (i.e. an upper bound on the Hausdorff dimension) which may render this identification unlikely. Nevertheless, we find certain qualitative features which would be expected of a $c>1$ theory. For instance, in addition to the positive susceptibility exponent, we find that whereas in the $c=1$ case the density of states itself diverges as one approaches the critical point, in the D-matrix models various derivatives of the density of states diverge, with the order of the derivative depending on D. This qualitatively different behaviour of the density of states could be a signal of the conjectured "phase transition" at $c=1$.

1. Introduction

Matrix models have recently become popular as representations of two-dimensional quantum gravity coupled to matter. Various types of matrix models have been studied, and their correspondence with certain kinds of matter coupled to 2D gravity elucidated. One of these is the hermitian one-matrix model whose generic critical behaviour is that of non-unitary matter coupled to gravity [1]. A generalization is the case of multi-matrix models, the matter content of which is identified as unitary with $c<1$ [2,3]. Another example is the case of a hermitian one-matrix model, where the matrix is a function of a single continuous parameter, which has been identified with a model of gravity coupled to a single boson ($c=1$) [4–7]. In the former case attempts have been made to construct $c>1$ models [8]. In the search for a matrix model representation of $c>1$ matter it is more natural to work in the context of the latter (i.e., time-dependent matrices), since one expects to increase the value of c by increasing the number of matrices.

Another motivation is to look for different kinds of critical behaviour. In the one-matrix case universality is a consequence of the fact that the critical behaviour is determined only by the behaviour of the potential near its critical point. In the multi-matrix examples that we shall consider here, the potential is a function of several variables, whose behaviour near the critical point is the topic of singularity theory [9]. Given the fact that there exist different types of singularities (e.g., A, D, E), one would correspondingly expect different universality classes.

Before proceeding to a review of the $c=1$ case in detail, let us briefly mention the general strategy adopted in refs. [4–7]. There the evaluation of the free energy of the matrix model was reduced to the calculation of the ground state energy of a system of N non-interacting fermions in a one-dimensional potential. In the $N\to\infty$ limit

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(planar limit), the scaling law for the free energy about the critical point of the potential allows for the computation of the string susceptibility exponent, which was found to be zero. The mass-gap exponent ν_{string} was found to be bounded from below by zero. Numerical results were quoted to argue that this bound is saturated, i.e. that $\nu_{\text{string}}=0$. Both of these results are consistent (up to logarithmic factors) with the continuum KPZ analysis [10].

To compute the free energy of the D-matrix model, we find it necessary to appropriately constrain the matrix integration measure, after which the evaluation of the free energy of this constrained model reduces to the calculation of the ground state energy of a system of N non-interacting fermions (bosons) in D dimensions for $D=$odd (even). Since the ground state energy of a gas of non-interacting bosons is identically zero, the constrained even-D models are trivial, and we shall henceforth consider only odd values of D. Here, the potential can depend on anywhere from one to D variables; defining this number as v, we show that for odd D and v, the scaling laws for the free energy result in positive susceptibility exponents. However, the lower-bound on ν_{string} that we find, makes the identification of these models with those of string theory difficult. Nevertheless, as we shall see, our models are not entirely fruitless.

In the next section we review the $c=1$ case in detail. Section 3 deals with the reduction of the D-matrix model to a gas picture. The scaling laws for the free energy in the planar limit are obtained in section 4. The mass-gap exponent (inverse of Hausdorff dimension) is also treated here. We conclude with a discussion section in which we attempt an interpretation of our constrained matrix models. The different qualitative behaviour of the density of states, in contrast to the $c=1$ model, is interpreted as characterizing a "phase transition" at $c=1$.

2. $c=1$ reviewed

The partition function for a one-dimensional hermitian one-matrix model can be written as

$$Z=\int \mathscr{D}\Phi(t)\,\exp\left(-\frac{N}{g}\int \mathrm{d}t\,\mathrm{Tr}[\dot{\Phi}^2(t)+V(\Phi)]\right), \tag{1}$$

where g is related to the cosmological constant. The evaluation of the free energy of this model reduces to the computation of the ground state energy of N fermions at zero temperature, with the hamiltonian [4–7]

$$H=\sum_{i=1}^{N}\left(-\frac{1}{2N}\frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2}+\frac{N}{g}V(\lambda_i\sqrt{g})\right),$$

where λ_i, $i=1,\ldots,N$ are the N eigenvalues of the matrix Φ. The ground state energy of this non-interacting system of N fermions is simply the sum of the first N eigenvalues of the one-body hamiltonian:

$$E_0=\sum_{i=1}^{N} e_i\,.$$

Defining a rescaled energy $\varepsilon_i=e_i g/N$, the normalization condition for the total number of particles

$$N=\int_0^{e_{\mathrm{F}}}\rho(e)\,\mathrm{d}e\,,$$

becomes

$$g=\int_0^{\varepsilon_{\mathrm{F}}}\rho(\varepsilon)\,\mathrm{d}\varepsilon\,, \tag{2}$$

where ε_{F} is the rescaled Fermi energy, and up to overall constants, one gets

$$\rho(\varepsilon)=\int \frac{1}{\sqrt{\varepsilon-V(\lambda)}}\,\mathrm{d}\lambda\,, \tag{3}$$

where the integral is between the turning points of V. Eq. (2) implies that

$$\mathrm{d}g/\mathrm{d}\varepsilon_{\mathrm{F}}=\rho(\varepsilon_{\mathrm{F}})\,. \tag{4}$$

The singularity of the integral arises only from the behaviour of the potential near its critical point, where one can Taylor expand the potential as

$$V(\lambda)=\varepsilon_{\mathrm{c}}-(\lambda-\lambda_{\mathrm{c}})^2+\ldots\,,$$

where $\varepsilon_{\mathrm{c}}\equiv V(\lambda_{\mathrm{c}})$ and $V'(\lambda_{\mathrm{c}})=0$. The dominant contribution to the integral (3) can be evaluated as

$$\rho(\varepsilon_{\mathrm{F}})=\int_{\sqrt{\eta}}^{c}\frac{1}{\sqrt{x^2-\eta}}\,\mathrm{d}x\,, \tag{5}$$

where $\eta\equiv\varepsilon_{\mathrm{c}}-\varepsilon_{\mathrm{F}}$, $x\equiv\lambda-\lambda_{\mathrm{c}}$, and the upper limit c is irrelevant, for its contribution to the integral is regular as $\varepsilon_{\mathrm{F}}\to\varepsilon_{\mathrm{c}}$. Then the dominant contribution to ρ is

$$\rho(\eta)\sim\log\eta\,,$$

from which with eq. (4) we see that

$$g-g_c\sim\eta\log\eta\,. \tag{6}$$

The string susceptibility exponent is read off from

$$\eta\sim(g-g_{\mathrm{c}})^{1-\gamma}\,,$$

so from eq. (6) one sees that up to logarithmic factors, $\gamma=0$.

To obtain the susceptibility exponent, it is sufficient to consider only the planar limit (zeroth order WKB), which is implicit in the above analysis. For potentials with kth order maxima, a similar analysis [6] yields $\gamma=-(k-2)/(k+2)$. We note that, regardless of the order of criticality in these models, $\gamma\leqslant 0$.

The mass-gap exponent is found from $\langle x^2\rangle\sim\langle n\rangle^{2\nu}\geqslant\rho^2$, where $\langle n\rangle$ is identified with the "area" of the surface (i.e. number of vertices in the graph). Since $g-g_{\mathrm{c}}\sim 1/\langle n\rangle$, then $\langle x^2\rangle\geqslant(\log\langle n\rangle)^2$. This means that modulo the logarithm, $\nu\geqslant 0$. In ref. [5] it was argued, based on numerical results, that the angular excitations seem not to affect the lowest excitation (as in the hydrogen atom), and hence $\nu=0$.

3. D-matrix models

Here we shall show that the vacuum energy of a D-matrix model (with the matrices $\Phi^K(t)$, $K=1,\ldots,D$) can be found as the ground state energy of an N-body hamiltonian in D dimensions, where the hamiltonian describes fermions (bosons) for D odd (even).

Starting from a generalization of eq. (1) to the D-matrix case, the corresponding hamiltonian describing the quantum mechanics of D matrices is [#1]

$$\tilde{H}=-\frac{1}{2N^{1/D}}\sum_{K=1}^{D}\sum_{i,j=1}^{N}\left(\frac{\partial}{\partial\Phi_{ij}}\right)^2+\left(\frac{N}{g}\right)^{1/D}\mathrm{Tr}\,V(g^{1/2D}\Phi^K)\,,$$

[#1] Although there is some freedom in choosing the coefficients (i.e. the powers of g and N) in this equation, the choice is partially dictated upon us – see footnote 3.

where the potential in general depends on all the matrices Φ^K. Following ref. [5], the ground state energy can be obtained from the variational principle

$$E=\min_{\psi}\left(\frac{(\psi\tilde{H}\psi)}{(\psi\psi)}\right), \tag{7}$$

where

$$(\psi\tilde{H}\psi)=\int\prod_{K=1}^{D}\prod_{i,j}\mathrm{d}\Phi_{ij}^{K}\left[\frac{1}{2N^{1/D}}\sum_{K=1}^{D}\sum_{i,j}\left(\frac{\partial\psi}{\partial\Phi_{ij}^{K}}\right)^2+\psi^2\left(\frac{N}{g}\right)^{1/D}\mathrm{Tr}\,V\right],$$

and

$$(\psi\psi)=\int\prod_{K=1}^{D}\prod_{i,j}\mathrm{d}\Phi_{ij}^{K}\,\psi^2\,.$$

As in ref. [5], this leads to

$$E=\min_{\chi}\left(\frac{(\chi H\chi)}{(\chi\chi)}\right), \tag{8}$$

with

$$\chi(\lambda^1,\ldots,\lambda^D)=\Delta(\lambda^1)\ldots\Delta(\lambda^D)\psi(\lambda^1,\ldots,\lambda^D)\,,$$

where $\Delta(\lambda_i^K)$ is the Vandermonde determinant of the matrix Φ^K and

$$H=\sum_{i=1}^{N}\left[-\frac{1}{2N^{1/D}}\sum_{K=1}^{D}\left(\frac{\partial}{\partial\lambda_i^K}\right)^2+\left(\frac{N}{g}\right)^{1/D}V(g^{1/2D}\lambda_i)\right]. \tag{9}$$

The intermediate steps involved in going from (7) to (9) are similar to those of ref. [5], but here we outline the steps to illustrate the role of the Vandermonde determinants corresponding to the D matrices. Upon diagonalizing all the D matrices with the same unitary transformation (see below), $\Phi^K=U^{\dagger}\Lambda^K U$, where Λ^K is diagonal in λ_i^K, we have

$$\frac{(\psi\tilde{H}\psi)}{(\psi\psi)}=\frac{1}{\int\prod_K\mathrm{d}(U)_{\mathrm{U}(N)}\prod_i\mathrm{d}\lambda_i^K\,\Delta^2(\lambda^K)\psi^2}\int\prod_{K=1}^{D}\mathrm{d}(U)_{\mathrm{U}(N)}\,\Delta^2(\lambda^K)\prod_{j=1}^{N}\mathrm{d}\lambda_j^K$$
$$\times\left[\frac{1}{2N^{1/D}}\sum_{K=1}^{D}\sum_{i=1}^{N}\left(\frac{\partial\psi}{\partial\lambda_i^K}\right)^2+\psi^2\left(\frac{N}{g}\right)^{1/D}\sum_{i=1}^{N}V(g^{1/2D}\lambda_i^K)\right].$$

In writing the Tr $V(\Phi^K)$ term as $\sum_{i=1}^{N}\mathrm{V}(\lambda_i^K)$ one must diagonalize all the D matrices simultaneously, which simply means that they must all commute. This constraint amounts to the angular parts of all the matrices being equal. This, in addition to the form of the kinetic term, also motivates the consistent ansatz that the wave function ψ may be a function of the eigenvalues only (see the paragraph below). As a result, since the arguments of the integrals are independent of the angular variables, the factor $\int\mathrm{d}(U)_{\mathrm{U}(N)}$ in the numerator cancels the same factor in the denominator. Then, upon absorbing the Vandermonde determinants into the ψ's, and partially integrating, we get

$$\frac{(\psi\tilde{H}\psi)}{(\psi\psi)} = \frac{1}{\int\prod_K\prod_j \mathrm{d}\lambda_j^K\chi^2}\int\prod_{K=1}^{D}\prod_{j=1}^{N}\mathrm{d}\lambda_j^K\left[\sum_{K=1}^{D}\sum_{i=1}^{N}\frac{1}{2N^{1/D}}\left(\frac{\partial\chi}{\partial\lambda_i^K}-\chi\frac{\partial\log\Delta(\lambda_i^K)}{\partial\lambda_i^K}\right)^2+\chi^2\left(\frac{N}{g}\right)^{1/D}\sum_{i=1}^{N}V(g^{1/2D}\lambda_i^K)\right]$$

$$= \frac{1}{(\chi\chi)}\int\prod_{K=1}^{D}\prod_{j=1}^{N}\mathrm{d}\lambda_j^K$$

$$\times\sum_{i=1}^{N}\left(\sum_{K=1}^{D}\frac{1}{2N^{1/D}}\left\{-\chi\frac{\partial^2\chi}{\partial\lambda_i^{K2}}-\chi^2\left[\frac{\partial^2\log\Delta(\lambda_i^K)}{\partial\lambda_i^{K2}}+\left(\frac{\partial\log\Delta(\lambda_i^K)}{\partial\lambda_i^K}\right)^2\right]\right\}+\chi^2\left(\frac{N}{g}\right)^{1/D}V(g^{1/2D}\lambda_i^K)\right).$$

The two terms involving the log's cancel identically, leaving us with eqs. (8) and (9), as promised.

Here λ_i^K, $i=1, ..., N$, is the set of eigenvalues of Φ^K; we note that we can define the D-vectors $\boldsymbol{\lambda}_i=(\lambda_i^1, ..., \lambda_i^D)$. The ansatz we have chosen is that the wave function ψ is a singlet of D copies of a *single* $\mathrm{U}(N)$, where the $\mathrm{U}(N)$ corresponds to the angular part of each matrix. As a result, the symmetry of the resulting theory is $\mathrm{U}(N)$ and not $\mathrm{U}(N)^D$. In other words, we require the angular parts of all the matrices to be equal, and hence there is a *single* $\mathrm{U}(N)$ that acts on all the D matrices [#2]. This is possible only if all the matrices commute with each other. It is important to note that this places a strong contrast on the interactions. This makes the identification of our models with a theory of 2D gravity non-trivial. Nevertheless, we shall elaborate on a possible interpretation in section 5. The wave function $\psi(\boldsymbol{\lambda}_i)$, $i=1, ..., N$, is symmetric under exchange of any two "coordinate" vectors $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_j$ since such a permutation is a $\mathrm{U}(N)$ transformation, and we have chosen a singlet wave function. Under such an exchange, due to the presence of the Vandermonde determinants, $\chi(\boldsymbol{\lambda}_1, ..., \boldsymbol{\lambda}_N)$ is clearly an antisymmetric (symmetric) wave function for D odd (even), and so the hamiltonian, eq. (9), correspondingly describes a system of N non-interacting fermions (bosons) in D dimensions, in an external field described by the potential V. In the next section we will apply statistical mechanical techniques to this gas of particles to derive the scaling laws for the D-matrix models.

In the path integral formulation (see Brézin et al. [7]), the Fermi statistics enters through the appearance, in the measure, of a product of Vandermonde determinants of all the matrices, of the form

$$\prod_{K=1}^{D}\Delta(\Phi^K(0))\Delta(\Phi^K(T)),$$

where the integration of the "time" variable runs from zero to T. This leads to an antisymmetrization of the final states with respect to the initial states.

4. Scaling laws

This section contains the derivation of the scaling laws near the critical point and the corresponding susceptibility and mass-gap exponents.

We begin by reviewing the standard statistical mechanical analysis of a D-dimensional gas of particles (fermions or bosons) in an external field. To obtain the scaling laws, one needs the expression for the density of states which one obtains from

$$N=\sum_i n_i,$$

where n_i is the occupation number of the ith state. In the large-N limit, one writes

$$N=\int\frac{1}{h^D}\mathrm{d}\Gamma n(e(p,q)), \tag{10}$$

[#2] Note that while the wave function does have the larger symmetry of $\mathrm{U}(N)^D$, the hamiltonian does not; the symmetry group in the theory is only $\mathrm{U}(N)$.

where $d\Gamma$ is the volume element in phase space, $d\Gamma = d^D q\, d^D p$, h is Planck's constant, and $e(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{p}^2/2m + U(\boldsymbol{q})$ is the classical energy. $n(e)$ is the appropriate equilibrium distribution, at fixed temperature, depending on whether the particles are fermions or bosons. Since $e(\boldsymbol{p}, \boldsymbol{q})$ depends only on the magnitude p of $\boldsymbol{p}$, we have $d^D p \propto p^{D-1} dp = m[2m(e-U)]^{(D-2)/2} de$. Since we are interested in the ground state energy of this gas of particles, we consider the zero temperature distribution. For bosons all the particles are at $e=0$, resulting in a zero total energy. Since we have shown in the previous section that the even-D matrix models correspond to a gas of bosons, we see that these models do not possess non-trivial scaling laws. Henceforth, therefore, we shall specialize to the odd-D matrix models, i.e., a gas of fermions, where $n(e) = 1$ for $e \leqslant e_F$, and zero otherwise, where e_F is the Fermi energy. Then eq. (10) can be written as

$$N \propto \left(\frac{m}{\hbar^2}\right)^{D/2} \int_0^{e_F} de \int_\Omega d^D q [e - U(\boldsymbol{q})]^{(D-2)/2},$$

where Ω is the classically allowed spatial region (for $D=1$, this would be the region between the turning points). The density of states is found from

$$N = \int_0^{e_F} \rho(e) de, \tag{11}$$

which gives

$$\rho(e) \propto \left(\frac{m}{\hbar^2}\right)^{D/2} \int_\Omega d^D q [e - U(\boldsymbol{q})]^{(D-2)/2}. \tag{12}$$

The ground state energy of the matrix hamiltonian (9) is the sum of the first N eigenvalues of the single particle hamiltonian

$$H(\boldsymbol{\lambda}) = -\frac{1}{2N^{1/D}} \frac{\partial^2}{\partial \boldsymbol{\lambda}^2} + \left(\frac{N}{g}\right)^{1/D} V(g^{1/2D} \boldsymbol{\lambda}).$$

Comparing with $e = -(\hbar^2/2m)\partial^2/\partial q^2 + U(\boldsymbol{q})$, we identify $(m/\hbar^2)$ with $N^{1/D}$. To eliminate N, and to make the g-dependence explicit, we perform the rescaling [#3]

$$e \to \varepsilon = \left(\frac{g}{N}\right)^{1/D} e.$$

Eqs. (11) and (12) then become

$$g = \int_0^{\varepsilon_F} \rho(\varepsilon) d\varepsilon, \tag{13}$$

where ε_F is the rescaled Fermi energy, and up to overall constants,

$$\rho(\varepsilon) = \int_\Omega d^D \lambda [\varepsilon - V(\boldsymbol{\lambda})]^{(D-2)/2}, \tag{14}$$

[#3] The elimination of N from (11), and the finiteness of ε (i.e. renormalized energy) as $N \to \infty$, allows for the replacement of $(1/D)$ with any other positive power which is less than, or equal to, $(1/D)$; the choice of $1/D$ is one of convenience. However, having equal powers of N in the kinetic and the potential term is not by choice. This has an important ramification regarding the existence of a topological expansion in these models – see section 5.

where $\lambda \equiv g^{1/2D} q$.

All quantities of interest can be obtained from the density of states; for instance one can rewrite (13) as

$$\rho(\varepsilon_F) = \frac{dg}{d\varepsilon_F},$$

from which one then obtains ε_F as a function of g: $\varepsilon_F(g)$. In turn, we define the rescaled ground state energy as

$$E = \frac{1}{N^{1+1/D}} \sum_{k=1}^{N} e_k,$$

which in the large-N limit becomes, in terms of ε,

$$E = \frac{1}{g^{1+1/D}} \int_0^g \varepsilon(t) dt,$$

with $t = gk/N$, giving

$$\frac{\partial}{\partial g} (g^{1+1/D} E) = \varepsilon_F(g).$$

From this equation one can get the scaling law for E in terms of that of ε_F, as g approaches its critical value. The dominant part of the specific heat $C \propto \partial^2 E / \partial g^2$ can be obtained from $C \sim \varepsilon_F'(g) \sim \rho^{-1}(\varepsilon_F(g))$. Since $g - g_c \sim 1/\langle n \rangle$, where $\langle n \rangle$ is the mean number of vertices in the graph, if our models are to be identified with a world "surface" embedded in some space-time, then the Hausdorff dimensions $(1/\nu)$ of these "surfaces" can be found from $\langle x^2 \rangle \sim \langle n \rangle^{2\nu} \geqslant \rho^2$.

For the case when $D = 1$, eq. (14) reduces to eq. (3), from which we recover the $c = 1$ scaling law

$$\rho_{D=1}(\eta) \sim \log \eta,$$

i.e. $\gamma = 0$ and $\nu = 0$ (up to logarithms).

For $D \geqslant 3$, the potential is a function of many variables whose number, v, varies anywhere from 1 to D. As we will see below, the scaling laws depend on both v and D. Since the scaling laws are determined primarily by the behaviour of the potential near its critical points, we appeal to singularity theory to provide us with the particular form of such near-critical multivariable potentials. Although singularity theory is concerned mainly with complex functions of complex variables, there does exist, among others, an ADE classification of real functions of real variables. In the case of hermitian matrix models, it is the latter that shall occupy our interest. However, here we will mostly deal with some of the A-type singularities, for there exists a rich variety of critical behaviour even for this restricted set. We may treat the details of the D- and the E-series, and also higher-modality singularities (where there exist marginal perturbations) elsewhere.

A remark about universality is in order. In the $D = 1$ case, where the potential is a function of only one variable, universality is a reflection of the fact that the potential near its (non-degenerate) critical point, can always be brought to a quadratic (or higher order, for the multicritical models) form by an appropriate coordinate transformation. Similarly, multi-variable functions can be brought to certain normal forms which are, in fact, the near-critical forms that the function can take, and these are classified by singularity theory. One would then expect a corresponding ADE-type classification of the universality classes of the multi-matrix models.

We now continue on to the derivation of the scaling laws and the exponents. We shall consider potentials, within the A-series, with rotational invariance, in which case we can rewrite (14) as

$$\rho(\varepsilon_F) = \int_\Omega dr\, r^{v-1} [\varepsilon_F - V(r)]^{(D-2)/2}, \tag{15}$$

where we have dropped overall constants coming from coordinates on which the potential does not depend. Here $r^2=\lambda_1^2+\lambda_2^2+...\lambda_v^2$, $1\leqslant v\leqslant D$, and Ω is the region between the turning points in the radial direction. It is helpful to keep in mind the following particular forms of the potential, without any loss of generality:

$$V(r)=r^2-r^4,$$

which is an upside-down rotated mexican-hat potential and

$$V(r)=(r^2-\tfrac{1}{2})^2,$$

which is a regular rotated mexican-hat potential. The corresponding maximum critical points, r_c, and critical values, ε_c, are $(1/\sqrt{2}, \tfrac{1}{4})$ and $(0, \tfrac{1}{4})$, respectively. We emphasize again that these are only examples that aid visualization; what follows holds in complete generality.

The scaling laws correspond to the behaviour of $\rho(\varepsilon_F)$ as ε_F approaches the critical value of the potential, ε_c. In contrast to the $D=1$ ($c=1$) case, where ρ itself is singular in this limit, eq. (14) does not behave in a singular manner for $D\geqslant 3$. However, we can still extract the dominant contribution to ρ in this limit by considering various derivatives of $\rho(\varepsilon_F)$, with the order of the derivative depending on D and v in general. Upon differentiating $\rho(\varepsilon_F)$ an appropriate number of times and Taylor-expanding the potential, it can be seen that the dominant contribution comes from the leading term in this expansion; to this order, the singular part comes from the near-critical limit of the integral.

For potentials of the first type, differentiating the density of states $n=(D-1)/2$ times gives

$$\rho^{(n)}(\varepsilon_F)=\int_{\Omega}\frac{r^{v-1}\mathrm{d}r}{\sqrt{\varepsilon_F-V(r)}}.$$

Near the critical point, we write $x=r-r_c$, in terms of which

$$\rho^{(n)}(\varepsilon_F)=\int_{-r_c}^{-\sqrt{\varepsilon_F-\varepsilon_c}}\frac{(x+r_c)^{v-1}\mathrm{d}x}{\sqrt{\varepsilon_F-\varepsilon_c+x^2}}.$$

The dominant contribution is from the r_c^{v-1} term (if $r_c\neq 0$) in the numerator and comes from the upper limit of the integral, giving

$$\rho^{(n)}(\eta)\sim\log\eta,$$

where $\eta\equiv\varepsilon_c-\varepsilon_F$. Upon integrating n times, this becomes

$$\rho(\eta)\sim\eta^{(D-1)/2}\log\eta, \tag{16}$$

since any non-zero integration constants that might have appeared are less dominant. We note that this result is independent of the number of variables in the potential, v.

In the case when $r_c=0$, which is the case corresponding to potentials of the second type, above, then the dominant contribution can be extracted from the mth derivative of ρ, where $m=(D+v-2)/2$, for odd v:

$$\rho^{(m)}(\eta)=\int_{\sqrt{\eta}}^{r_0}\frac{x^{v-1}\mathrm{d}x}{\sqrt{x^2-\eta}}, \tag{17}$$

where r_0 is the turning point at the outer rim of the potential. Since r_0 is not a critical point of the potential, it only contributes a regular term to the integral, and hence the singular part comes from the lower limit, giving

$$\rho^{(m)}(\eta)\sim\log\eta. \tag{18}$$

This equation has been derived from (17) by partial integration, for v odd. For v=even, no singularity results, as can be seen by integration by parts. Hence (18) gives, for odd v,

$$\rho(\eta)\sim\eta^{(D+v-2)/2}\log\eta\,. \tag{19}$$

Qualitatively, the reason for the v-dependence in this equation, in contrast to eq. (16), can be attributed to the difference in the manner in which criticality is approached, between the two afore-mentioned types of potentials. For instance, in the upside-down rotated mexican-hat potential, the Fermi sea spills over the upper rim of the potential, whereas in the case of the regular mexican hat, no spilling occurs, since the region above ε_c is also bound.

Up to now, we have considered only the specific case of the A_1^+ type of singularity, i.e., potentials which take the form $V=x_1^2+x_2^2+\ldots+x_v^2$ near the critical point. In the case of real singularities (potentials), there is also the possibility of having a mixture of + and − signs in front of the various terms. By considering such potentials with hyperbolic symmetry, one can still define a "radial" coordinate, after which the calculation proceeds in exactly the same manner as in the A_1^+ case, hence yielding the same scaling laws.

Remaining within the set of rotationally invariant potentials, we can explore other types of critical behaviour. For s-critical potentials, where $s-1$ derivatives of $V(r)$ vanish at the critical point, the dominant part of ρ can be found directly from

$$\rho(\eta)\sim\int_{\eta^{1/s}}^{r_c}\mathrm{d}x\,(x+r_c)^{v-1}(x^s-\eta)^{(D-2)/2}\,.$$

For $D=1$, this reproduces the result of ref. [6], i.e.,

$$\rho(\eta)_{D=1}\sim\int_{-\eta^{1/s}}^{r_c}\frac{\mathrm{d}x}{\sqrt{x^s-\eta}}\sim\eta^{1/s-1/2}\,,$$

for $s>2$. Here $v\geqslant(s-2)/(s+2)$. Further, we can consider the case of $D=3$, $v=1$; in that case, the identity

$$\left(1+\frac{2}{s}\right)\int\mathrm{d}x\,\sqrt{x^s-\eta}=\frac{2}{s}x\sqrt{x^s-\eta}-\eta\int\frac{\mathrm{d}x}{\sqrt{x^s-\eta}}$$

tells us that

$$\rho(\eta)\sim\eta^{1/s+1/2}\,, \tag{20}$$

and $v\geqslant-(s+2)/(s+3)$. We mention in passing that the same result also holds for $v=2$ and $v=3$ when $s=3$, and for $v=2$ when $s=4$. These cases are sufficient to provide us with examples of models with positive susceptibility, as we shall see below.

The definition of the string susceptibility exponent, γ, can be written in terms of the scaling of the density of states with the Fermi energy as

$$\rho(\eta)\sim\eta^{\gamma/(1-\gamma)}\,.$$

Therefore, up to the logarithms which we shall discuss below, eqs. (16) and (19) give

$$\gamma=(D-1)/(D+1)\,,\quad\gamma=1-2/(D+v)\,,$$

for the upside-down and regular rotated mexican-hat potentials, respectively. Observe that the values of γ for the two types of potentials are identical. This is, again, a manifestation of universality. From eq. (20), we see that our examples of $D=3$ multi-critical models not only do not suffer from logarithmic corrections, but also have positive susceptibility exponents

$$\gamma=(s+2)/(3s+2)\,,$$

in contrast to the $c=1$ ($D=1$) case [6]. We note that all the $D>1$ models have strictly positive susceptibility exponents, and negative lower bounds on the ν's. Assuming that the angular contributions do not affect the lowest excitation (i.e. that the inequality is saturated), as in the $D=1$ case, the "surfaces" embedded in space–time would end-up having a negative Hausdorff dimension. This suggests that we may not be dealing with a string moving in a space–time, or a model of 2D gravity coupled to $c>1$ matter; however, we are still able to use these models as a laboratory for studying certain qualitative features which are expected of the latter.

5. Discussion

As we promised in section 3, in addition to a general discussion, we will now try to put forth an interpretation of the constraint that we placed on the angular variables of the matrices, namely, that they be all equal. Of course this is a strong constraint which further obfuscates an interpretation of our models in terms of known theories of matter coupled to 2D gravity.

In the hamiltonian picture used here, this constraint arises from our requirement that all the matrices be simultaneously diagonalizable. This was necessary in order to write (7) as (8), which is equivalent to the fact that our wave function is a singlet of D copies of a single $\mathrm{U}(N)$.

One way of imposing this constraint, at the path-integral level, is to introduce a delta-function into the integration measure which equates the angular parts of all the matrices. However, it is more transparent to impose the same constraint in a somewhat different manner as delta-functions involving the commutator of the matrices themselves. This has the advantage that it introduces an interaction between the matrices when raised into the exponent (action). For example, we can formally write [#4] $\delta([A, B])$ as $\int \mathrm{d}\Lambda \exp \Lambda \operatorname{tr}([A, B])^2$, where A and B are any two matrices, and Λ is a Lagrange multiplier. This would introduce interactions of the type $\operatorname{tr}(ABAB)$ and $\operatorname{tr}(AABB)$ in the action. These vertices resemble the four-vertices appearing in the matrix-model representation of the Ising model [3]. The consequence is that we are dealing with an Ising-like model where the spins carry additional quantum numbers. It is important to study interactions of this type in order to better understand the matter content of this theory.

$\gamma>0$ models have been obtained in the context of "time"-independent one-matrix models; however, those were identified with theories of matter coupled to several world sheets touching each other [8]. This interpretation was a consequence of $(\operatorname{tr} A^2)^2$-type interactions in the lagrangian. In contrast, the absence of such interactions in our models (see the single trace in the hamiltonian, above eq. (7)) implies that we are dealing with a single object (perhaps not a string) coupled to $c>1$ matter. The fact that there is no overlap between our values of γ and those of ref. [8] (with the possible exception of $\gamma=\frac{1}{2}$) also means that our models are distinct from those of ref. [8]. It is to be understood that at the level of the partition function, $\gamma>0$ is only indicative of $c>1$ matter coupled to world sheet; there is numerical evidence that for $c>1$, $\gamma>0$ and $\nu>0$ [11].

A few words about the presence of the logarithms in our scaling laws: In the $D=1$ ($c=1$) case, this has been linked to the presence of a tachyon which happens to be massless in one dimension [12]. The logarithms that we find would also appear to indicate that we are still at $c=1$; however, we point out that the $c=-2$ theory [13] also has such logarithms. This calls for a better understanding of the origin of these logarithms. In the somewhat different context of topological gravity, albeit with $c=1$ features, the logarithm has also been attributed to the $\mathrm{SL}(2, \mathbb{C})$ invariance of the sphere [14]. There the free energy was expanded in powers of the cosmological constant in such a way that the coefficients were identified with the (virtual) Euler characteristic of the moduli space of Riemann surfaces with punctures. However, it is not clear if such an expansion is allowed in the $c>1$ region. The logarithmic behaviour of $\langle x^2 \rangle$, which is of the same origin as that of ρ, however, is still typical of smooth or Liouville surfaces [11], since this is predicted by Liouville perturbation theory [15].

[#4] We thank I. Kostov for suggesting this particular way of writing the constraint.

Let us mention, in passing, that the $1/N$ expansion in the quantum mechanical D-matrix models, treated here, does correspond to a topological expansion. Although, at first sight, counting the contribution of the propagators and the vertices in the graphs (dual to the triangulation) would suggest a total contribution of $[N^{(1/D)}]^{\chi}$, where χ is the Euler character of the "surface" being triangulated, this is spoiled by the fact that each face in the graph still contributes an N, and not $N^{(1/D)}$. It is easy to see from the hamiltonian, however, that there is a rescaling of the matrices (consistent with the criteria mentioned in footnote 3) and the coupling g which does resurrect the $1/N$ expansion as a topological one.

It is often conjectured that some sort of a "phase transition" is to occur at $c=1$. Our models allow us to give a slightly more accurate meaning to this expectation. Specifically, whereas in the $D=1$ ($c=1$) theory the density of states itself diverges at criticality, in the $D>1$ models nth order derivatives of ρ diverge, where n depends on D (see (16) and (19)). This is reminiscent of the usual definition of a phase transition.

We end by recapitulating some of the main features of these models which we have presented and, to various degrees, supported:

(1) The existence of an ADE classification of the universality classes.

(2) The existence of a constraint which identifies the D-matrix models with a D-dimensional gas of fermions/bosons.

(3) That the even-D models portray no scaling behaviour.

(4) That there are some $c \geqslant 1$ features; e.g. positive susceptibility, a logarithmic growth of $\langle x^2 \rangle$ with "area", and a behaviour of the density of states which is reminiscent of a phase transition at $c=1$.

(5) That there are also some stringy and some non-stringy features, e.g. the existence of a topological expansion, and a negative Hausdorff dimension (if the angular excitations are ignored), respectively.

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