sion of x on y, the line of regression of y on x, and the correlation coefficient. Is the correlation significant?

22. Following are two sets of pairs of observations on variables x and y:

x	y	x	y
1	2	1	2
2	3	2	3
3	3	3	4
4	2	4	5
5	4	5	5

Determine whether either of these sets exhibits a significant correlation between x and y.

APPENDIX A
SUMMARY OF FORMULAS

Following is a summary of important and useful formulas which have been developed in the text. The numbered equations are given the same numbers as in the text to facilitate reference to appropriate parts of the text.

Approximations

If a quantity Q is determined from quantities $a, b, \ldots$ by a relation $Q = f(a, b, \ldots)$, then the change ΔQ of the quantity produced by changes $\Delta a, \Delta b, \ldots$ is

$$\Delta Q = \frac{\partial Q}{\partial a}\Delta a + \frac{\partial Q}{\partial b}\Delta b + \frac{\partial Q}{\partial c}\Delta c + \cdots \qquad (2.8)$$

The Mean and Dispersion

The *mean* (or arithmetic mean or average) of a set of N numbers, of which a typical one is x_i, is

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad (3.2)$$

The *weighted mean* of a set of N numbers, of which a typical one is x_i with weight w_i, is

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \cdots + w_Nx_N}{w_1 + w_2 + \cdots + w_N} = \frac{\sum_{i=1}^{N} w_ix_i}{\sum_{i=1}^{N} w_i} \tag{3.3}$$

The *deviation* of a number x_i in a set of N numbers is

$$d_i = x_i - \bar{x} \tag{3.4}$$

The *mean deviation* of a set of N numbers x_i is

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} |d_i| = \frac{1}{N}\sum_{i=1}^{N} |x_i - \bar{x}| \tag{3.8}$$

The *standard deviation* of a set of N numbers x_i is

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N} d_i^2} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{x})^2} \tag{3.9}$$

The *variance* is defined as the square of the *standard deviation.*

Probability

If the probabilities for two independent events a and b are P_a and P_b, the probability for *both* to occur is the product P_aP_b. If events a and b are mutually exclusive, the probability for a *or* b to occur is the sum $P_a + P_b$.

Permutations and Combinations

The number of permutations of N objects is

$$N! = N(N-1)(N-2)(N-3)\cdots(4)(3)(2)(1) \tag{5.1}$$

The number of combinations of N objects, taken n at a time, is

$$C(N,n) = \frac{N!}{(N-n)!\,n!} = \binom{N}{n} \tag{5.3}$$

which also defines the binomial coefficients.

The *binomial theorem* for expansion of the binomial $(a+b)^N$ is

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^{N-n}b^n$$

$$= \sum_{n=0}^{N} \frac{N!}{(N-n)!\,n!} a^{N-n}b^n \tag{5.6}$$

The sum of the binomial coefficients for a given N is

$$(1+1)^N = 2^N = \sum_{n=0}^{N} \binom{N}{n} \tag{5.7}$$

Probability Distributions

The condition for a discrete probability distribution to be *normalized* is

$$\sum_n f(n) = 1 \tag{6.1}$$

The *mean* of a discrete probability distribution is

$$\bar{n} = \sum_n nf(n) \tag{6.2}$$

The *variance* of a discrete probability distribution is

$$\sigma^2 = \sum_n (n - \bar{n})^2 f(n) \tag{6.4}$$

The best estimate of the variance of a parent distribution, from a sample of this distribution, is

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{6.5}$$

Binomial Distribution

The *binomial distribution* gives the probability of n successes in N independent trials, if the probability of success in any one trial is p. The binomial distribution is given by

$$f_{N,p}(n) \doteq \binom{N}{n} p^n q^{N-n} \tag{7.1}$$

where $q = 1 - p$.

The *mean* of the binomial distribution is

$$\bar{n} = \sum_{n=0}^{N} n \binom{N}{n} p^n (1-p)^{N-n} = Np \tag{7.5}$$

The *variance* of the binomial distribution is

$$\sigma^2 = Np(1-p) = Npq \tag{7.7}$$

Poisson Distribution

The *Poisson distribution* is the limit of the binomial distribution as $N \to \infty$ and $p \to 0$ in such a way that the product $a = Np$ remains finite. The Poisson distribution is given by

$$f_a(n) = \frac{a^n e^{-a}}{n!} \tag{8.5}$$

The *mean* of the Poisson distribution is

$$\bar{n} = a \tag{8.6}$$

The *variance* of the Poisson distribution is

$$\sigma^2 = a. \tag{8.9}$$

Gauss Distribution

The *Gauss distribution*, or *normal error function*, is

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-m)^2} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m)^2/2\sigma^2} \tag{9.9, 9.16}$$

The *index of precision* h and the *variance* σ^2 are related by

$$\sigma^2 = \frac{1}{2h^2} \tag{9.15}$$

The *mean deviation* for the Gauss distribution is given by

$$\alpha = \frac{1}{\sqrt{\pi}\,h} = \sqrt{\frac{2}{\pi}}\,\sigma \tag{9.18, 9.19}$$

The probability for a measurement to fall within $T\sigma$ of the mean is

$$P(T) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{-t^2/2}\,dt \tag{9.22}$$

The Gauss distribution is approximately equal to the binomial distribution with the same mean and variance, if N is very large and p is finite. For very large N,

$$f_{N,p}(n) = \frac{N!}{(N-n)!\,n!} p^n q^{N-n}$$

$$\cong \frac{1}{\sqrt{2\pi Npq}} e^{-(n-Np)^2/2Npq} \tag{C.25}$$

Goodness of Fit

To compare a sample frequency $F(n)$ with a frequency $Nf(n)$ predicted by a parent distribution $f(n)$ for N trials, a suitable index of goodness of fit is

$$\chi^2 = \sum_n \frac{[Nf(n) - F(n)]^2}{Nf(n)} \qquad (11.1)$$

In using a table of values of χ^2, $\nu = K - r$, where K is the number of frequencies of the two distributions compared and r is the number of parameters of the parent distribution which are determined from the sample.

Standard Deviation of the Mean

The *standard deviation of the mean* $\bar{x}$ of a set of numbers x_i is

$$\sigma_m{}^2 = \frac{\sigma^2}{N} \qquad \text{or} \qquad \sigma_m = \frac{\sigma}{\sqrt{N}} \qquad (12.6)$$

if the numbers are distributed normally with parent standard deviation σ.

Propagation of Errors

If a quantity Q is determined from quantities $a, b, \ldots$ by a relation $Q = f(a, b, \ldots)$, the variance of the mean of Q is related to the variances of the means of $a, b, \ldots$ by

$$\sigma_{mQ}{}^2 = \left(\frac{\partial Q}{\partial a}\right)^2 \sigma_{ma}{}^2 + \left(\frac{\partial Q}{\partial b}\right)^2 \sigma_{mb}{}^2 + \cdots \qquad (13.8)$$

Method of Least Squares

If N observations are made on a quantity, and their errors are normally distributed, the *most probable value* of the quantity is

$$x = \frac{1}{N} \sum_{i=1}^{N} x_i \qquad (14.4)$$

and the *variance* of the most probable value is related to the variance of the individual observations by

$$\sigma_m{}^2 = \sum \frac{\sigma^2}{N^2} = \frac{\sigma^2}{N} \qquad (14.10)$$

If the observations x_i come from different parent distributions, characterized by their variances $\sigma_i{}^2$, then the most probable value of x is the weighted mean

$$x = \frac{\Sigma\, x_i/\sigma_i{}^2}{\Sigma\, 1/\sigma_i{}^2} \qquad (14.13)$$

and the *variance* of this weighted mean is given by

$$\frac{1}{\sigma_m{}^2} = \sum_i \frac{1}{\sigma_i{}^2} \qquad (14.16)$$

In an equation $y = mx + b$, the *most probable values* of m and b, from a set of pairs of observations (x_i, y_i) in which the x_i have no errors and all the y_i have errors belonging to the same distribution, are

$$m = \frac{N\, \Sigma\, x_i y_i - (\Sigma\, x_i)(\Sigma\, y_i)}{N\, \Sigma\, x_i{}^2 - (\Sigma\, x_i)^2}$$

$$b = \frac{(\Sigma\, y_i)(\Sigma\, x_i{}^2) - (\Sigma\, x_i y_i)(\Sigma\, x_i)}{N\, \Sigma\, x_i{}^2 - (\Sigma\, x_i)^2} \qquad (15.11)$$

The *variances* of the most probable values are given by

$$\sigma_m^2 = \frac{N\sigma^2}{\Delta} \tag{15.16}$$

and

$$\sigma_b^2 = \frac{\sigma^2 \Sigma x_i^2}{\Delta} \tag{15.17}$$

where

$$\Delta = N \Sigma x_i^2 - (\Sigma x_i)^2$$

and

$$\sigma^2 = \frac{1}{N} \sum (mx_i + b - y_i)^2 \tag{15.19}$$

where σ^2 is evaluated using the most probable values of m and b.

Correlations

The definition of the *linear correlation coefficient* r is

$$r = \sqrt{mm'} = \frac{N \Sigma xy - \Sigma x \Sigma y}{[N \Sigma x^2 - (\Sigma x)^2]^{1/2} [N \Sigma y^2 - (\Sigma y)^2]^{1/2}} \tag{16.8}$$

APPENDIX B

EVALUATION OF $\bar{n}$ AND σ FOR BINOMIAL DISTRIBUTION

To evaluate the sum

$$\bar{n} = \sum_{n=0}^{N} n \binom{N}{n} p^n (1-p)^{N-n} \tag{B.1}$$

we note that it is similar to an expression we have already encountered in considering the *normalization* of the binomial distribution, namely,

$$\sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} = 1 \tag{B.2}$$

The difference is that the sum in Eq. (B.1) contains an extra factor of n. But by means of a trick we can convert this into the form of the sum in Eq. (B.2).

From here on we drop the limits on the sums, as we did in Sec. 3, remembering always that n ranges from 0 to N. Now we differentiate both sides of Eq. (B.2) with respect to p, which is legitimate because the equation is true for all values of p between 0 and 1, as observed earlier. The advantage of doing this will appear shortly. Taking the derivative,

$$\sum \binom{N}{n} [np^{n-1}(1-p)^{N-n} - (N-n)p^n(1-p)^{N-n-1}] = 0 \tag{B.3}$$

This can be rewritten

$$\sum \binom{N}{n} np^{n-1}(1-p)^{N-n}$$

$$= \sum \binom{N}{n} (N-n)p^n(1-p)^{N-n-1}$$

$$= N \sum \binom{N}{n} p^n(1-p)^{N-n-1}$$

$$- \sum \binom{N}{n} np^n(1-p)^{N-n-1}$$

or

$$\sum n \binom{N}{n} [p^{n-1}(1-p)^{N-n} + p^n(1-p)^{N-n-1}]$$

$$= N \sum \binom{N}{n} p^n(1-p)^{N-n-1} \quad (B.4)$$

We now multiply both sides of the equation by $p(1-p)$:

$$\sum n \binom{N}{n} [(1-p)p^n(1-p)^{N-n} + pp^n(1-p)^{N-n}]$$

$$= Np \sum \binom{N}{n} p^n(1-p)^{N-n} \quad (B.5)$$

Combining the two terms on the left side, and using Eq. (B.2) in the right side

$$\sum n \binom{N}{n} p^n(1-p)^{N-n} = \sum nf_{N,p}(n) = Np \quad (B.6)$$

Now the left side of this expression is just our original expression for $\bar{n}$, Eq. (B.1); hence we conclude that

$$\bar{n} = Np \quad (B.7)$$

The calculation of σ^2 proceeds in a similar manner. The variance is given by Eq. (7.6), which we give again for convenience:

$$\sigma^2 = \Sigma\, (n - Np)^2 f_{N,p}(n) \quad (B.8)$$

To evaluate this sum we first rewrite Eq. (B.8) as

$$\sigma^2 = \Sigma\, (n^2 - 2nNp + N^2p^2) f_{N,p}(n)$$

$$= \Sigma\, n^2 f_N(n) - 2Np\, \Sigma\, nf_{N,p}(n) + N^2p^2\, \Sigma\, f_{N,p}(n) \quad (B.9)$$

The sums in the second and third terms are already known from Eqs. (B.6) and (B.2), respectively; using these, we find

$$\sigma^2 = \Sigma\, n^2 f_{N,p}(n) - (Np)^2 \quad (B.10)$$

To evaluate $\Sigma\, n^2 f_{N,p}(n)$ we differentiate Eq. (B.6):

$$\sum n \binom{N}{n} [np^{n-1}(1-p)^{N-n}$$

$$- (N-n)p^n(1-p)^{N-n-1}] = N \quad (B.11)$$

We multiply by $p(1-p)$ and rearrange terms as before, to obtain

$$\sum n^2 \binom{N}{n} p^n(1-p)^{N-n}$$

$$- Np \sum n \binom{N}{n} p^n(1-p)^{N-n} = Np(1-p) \quad (B.12)$$

Finally, using Eq. (B.6) again,

$$\sum n^2 \binom{N}{n} p^n (1-p)^{N-n} = (Np)^2 + Np(1-p) \tag{B.13}$$

or

$$\Sigma\, n^2 f_N(n) = Np(1 - p + Np) \tag{B.14}$$

Now, inserting this result into Eq. (B.10), we obtain

$$\sigma^2 = Np(1 - p + Np) - (Np)^2 = Np(1-p) = Npq \tag{B.15}$$

or

$$\sigma = \sqrt{Npq} \tag{B.16}$$

as stated in Sec. 7.

APPENDIX C
DERIVATION OF GAUSS DISTRIBUTION

Following is a derivation of the Gauss distribution function from some plausible assumptions. It is not intended as a substitute for empirical verification of this distribution, but as evidence that it *can* be derived from basic considerations. To be honest we must state that the mathematical derivation can be simplified considerably by making use of an approximation formula for factorials of large numbers (Stirling's formula). The use of this formula has been avoided here because, to a reader who is not familiar with its derivation, the development of the Gauss distribution using it is not likely to be very convincing.

We begin by assuming that the random error in a measurement is composed of a large number N of elementary errors, all of equal magnitude ϵ, and each equally likely to be positive or negative. With these assumptions, we can calculate the probability of occurrence of any particular error in the range $(-N\epsilon)$ to $(+N\epsilon)$. Having done this, we take the limit of this distribution as the number N becomes infinitely large and the magnitude ϵ infinitesimally small in such a way that the *standard deviation* of the distribution remains constant.

First, we note that the probability for n of the elementary errors to be positive and the other $N - n$ to be negative is given by the binomial distribution with $p = q = \frac{1}{2}$. The corresponding error, which we shall call y, is given by

$$y = n\epsilon - (N - n)\epsilon = (2n - N)\epsilon \tag{C.1}$$

The probability of occurrence of this particular error is

$$f_{N,1/2}(n) = \frac{N!}{(N-n)!\,n!\,2^N} \tag{C.2}$$

For future reference, we compute the standard deviation of y. Because each elementary error is as likely to be positive as negative, the mean value of y is zero. Therefore the standard deviation is given simply by

$$\sigma^2 = \Sigma\, y^2 f_{N,1/2}(n) = \Sigma\,(2n - N)^2\epsilon^2 f_{N,1/2}(n) \tag{C.3}$$

This sum is easily evaluated with the help of Eqs. (B.1), (B.2), and (B.14), setting $p = \frac{1}{2}$ in all these:

$$\begin{aligned}\sigma^2 &= 4\epsilon^2\,\Sigma\, n^2 f_N(n) - 4N\epsilon^2\,\Sigma\, n f_N(n) + N^2\epsilon^2\,\Sigma\, f_N(n)\\ &= 4\epsilon^2\frac{N}{2}\left(1 - \frac{1}{2} + \frac{N}{2}\right) - 4N\epsilon^2\frac{N}{2} + N^2\epsilon^2\\ &= \epsilon^2 N \end{aligned}\tag{C.4}$$

$$\sigma = \epsilon\sqrt{N} \tag{C.5}$$

To simplify notation in the following developments, we introduce a new index r, defined by the equation

$$2r = 2n - N \tag{C.6}$$

One immediate advantage of this change is suggested by Eq. (C.1), which now becomes simply

$$y = 2r\epsilon \tag{C.7}$$

Note that since the range of n is 0 to N, the range of r is from $-N/2$ to $+N/2$, in steps of unity. Furthermore, if N is *even*, r is always an *integer*, while if N is *odd*, r is always a *half-integer*. In either case, the quantities $N/2 + r$ and $N/2 - r$ which appear below are always integers.

We now express the probability for the error $y = 2r\epsilon$ in terms of the index r, using Eq. (C.6) in the form $n = r + N/2$. Equation (C.2) then becomes

$$f_{N,1/2}(r) = \frac{N!}{\left(\frac{N}{2} - r\right)!\left(\frac{N}{2} + r\right)!\,2^N} \tag{C.8}$$

The next larger possible value of the error y results from replacing r in Eq. (C.7) by $(r + 1)$. This error is then larger by an amount 2ϵ; so we call it $y + 2\epsilon$. The corresponding probability is obtained by inserting $(r + 1)$ for r in Eq. (3.8):

$$f_{N,1/2}(r+1) = \frac{N!}{\left(\frac{N}{2} - r - 1\right)!\left(\frac{N}{2} + r + 1\right)!\,2^N} \tag{C.9}$$

Thus if we call $f(y)$ the probability of occurrence of error y, we have

$$\begin{aligned} f(y) &= \frac{N!}{\left(\frac{N}{2} - r\right)!\left(\frac{N}{2} + r\right)!\,2^N}\\ f(y + 2\epsilon) &= \frac{N!}{\left(\frac{N}{2} - r - 1\right)!\left(\frac{N}{2} + r + 1\right)!\,2^N}\end{aligned}\tag{C.10}$$

These expressions are both rather complicated, but we note immediately that their *quotient* is fairly simple. That is,

$$\frac{f(y+2\epsilon)}{f(y)} = \frac{(N/2-r)!}{(N/2-r-1)!}\,\frac{(N/2+r)!}{(N/2+r+1)!}$$
$$= \frac{N/2-r}{N/2+r+1} \tag{C.11}$$

Next it is necessary to perform a somewhat tricky maneuver. Keeping in mind that we are eventually going to let $N \rightarrow \infty$ and $\epsilon \rightarrow 0$ at the same time, in such a way that the product $\sigma^2 = \epsilon^2 N$ remains constant, we now regard y as a continuous variable and $f(y)$ as a function of this variable. Because ϵ is small, we can approximate $f(y + 2\epsilon)$ as follows:

$$f(y+2\epsilon) \cong f(y) + 2\epsilon \frac{d}{dy} f(y) \tag{C.12}$$

Also, to facilitate taking the limit, it is convenient to express r and ϵ in terms of y, N, and σ, using Eqs. (C.5) and (C.7) as follows:

$$r = \frac{y}{2\epsilon} = \frac{y\sqrt{N}}{2\sigma} \qquad \epsilon = \frac{\sigma}{\sqrt{N}} \tag{C.13}$$

Inserting Eqs. (C.12) and (C.13) into Eq. (C.11),

$$\frac{f(y) + (2\sigma/\sqrt{N})f'(y)}{f(y)} = \frac{N/2 - y\sqrt{N}/2\sigma}{N/2 + y\sqrt{N}/2\sigma + 1} \tag{C.14}$$

where we have introduced the abbreviation $f' = df/dy$. Rearranging,

$$\frac{2\sigma}{\sqrt{N}}\frac{f'(y)}{f(y)} = \frac{N/2 - y\sqrt{N}/2\sigma}{N/2 + y\sqrt{N}/2\sigma + 1} - 1 \tag{C.15}$$

and

$$\frac{f'(y)}{f(y)} = -\frac{y/\sigma^2 + 1/\sqrt{N}\sigma}{1 + (y/\sqrt{N}\sigma + 2/N)} \tag{C.16}$$

Now, at last, we are ready to consider the limit of Eq. (C.16) as $N \rightarrow \infty$ while σ is constant. Clearly, in both numerator and denominator, the second term becomes very small compared to the first, if N is sufficiently large. So in the limit the terms containing $1/\sqrt{N}$ and $1/N$ both vanish, and we have simply

$$\frac{f'(y)}{f(y)} = -\frac{y}{\sigma^2} \tag{C.17}$$

This is a differential equation for the desired function $f(y)$; it is easily solved by noting that

$$\frac{f'(y)}{f(y)} = \frac{d}{dy}\ln f(y)$$

Making this substitution and integrating both sides of the equation, we find

$$\ln f(y) = -\frac{y^2}{2\sigma^2} + \text{const} \tag{C.18}$$

We represent the integration constant by ln A, where A is another constant, and take antilogs of both sides, to obtain

$$f(y) = Ae^{-y^2/2\sigma^2} \tag{C.19}$$

The value of the constant A is determined by recalling the interpretation of $f(y)$ discussed in Sec. 9. The quantity $f(y)\,dy$ is the probability that a single error will fall in the range y to $y + dy$. Since the total probability for the error to fall *somewhere* in the range of values of y (which is now, strictly speaking, $-\infty$ to $+\infty$) is unity, we must insist that

$$\int_{-\infty}^{\infty} Ae^{-y^2/2\sigma^2} = 1$$

or

$$A = \frac{1}{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}} \qquad \text{(C.20)}$$

Making the substitution $z = y/\sqrt{2}\sigma$, we obtain

$$A^{-1} = \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-z^2}\,dz \qquad \text{(C.21)}$$

The integral in this expression is evaluated in Appendix D and has the value $\sqrt{\pi}$. Hence

$$A = \frac{1}{\sqrt{2\pi}\sigma} \qquad \text{(C.22)}$$

and

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} \qquad \text{(C.23)}$$

Finally, we express the function in terms of the *observations* rather than their errors. If y is the error corresponding to an observation x, and the true value of the observed quantity is m, then $y = x - m$. In terms of x,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \qquad \text{(C.24)}$$

This form is usually called the Gauss distribution, or normal error function.

In the preceding discussion, the Gauss distribution has been shown to be an approximation of a distribution closely related to the binomial distribution, valid when the number N of independent events becomes very large. By similar methods it can be shown that *any* binomial distribution approaches the Gaussian form if N is very large and p is finite. Thus for large N we can represent a binomial distribution (which is very unwieldy for large N) by a Gauss distribution with the same mean and standard deviation as the binomial, namely, $m = Np$ and $\sigma = (Npq)^{1/2}$, respectively. Thus for large N we have approximately

$$\begin{aligned} f_{N,p}(n) &= \frac{N!}{(N-n)!\,n!} p^n q^{N-n} \\ &\cong \frac{1}{\sqrt{2\pi Npq}} e^{-(n-Np)^2/2Npq} \end{aligned} \qquad \text{(C.25)}$$

It should be noted that this is *not* a suitable approximation if p is extremely small. If p grows small as N grows large, then the appropriate approximations lead instead to the Poisson distribution, discussed in Sec. 8.

APPENDIX D

EVALUATION OF NORMAL ERROR INTEGRAL

In developing the Gauss distribution, it is necessary to know the value of the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

We denote the value of the integral by I. Then, since the variable of integration has no effect on the value of the result, we write

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \qquad \text{(D.1)}$$

Now Eq. (D.1) can also be interpreted as the double integral over the $x - y$ plane of the function

$$e^{-x^2}e^{-y^2} = e^{-(x^2+y^2)}$$

That is,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \qquad \text{(D.2)}$$

It may help to interpret this integral geometrically. Think of a tent whose floor is the $x - y$ plane and whose height above the $x - y$ plane at any point (x,y) is

$$e^{-(x^2+y^2)}$$

Then the integrand

$$e^{-(x^2+y^2)}\,dx\,dy$$

represents the volume of a column above the element of

floor area $dx\,dy$. Thus the quantity I^2 is just the total volume enclosed by the tent and its floor.

Now we transform Eq. (D.2) into polar coordinates, using as the element of floor area $dr\,(r\,d\theta)$ instead of $dx\,dy$, and using $r^2 = x^2 + y^2$. We thus obtain

$$I^2 = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} e^{-r^2}\,r\,dr\,d\theta \qquad \text{(D.3)}$$

This integral now can be expressed in terms of two integrals, each of which contains only one of the variables, as follows:

$$I^2 = \int_0^{\infty} e^{-r^2} r\,dr\left[\int_0^{2\pi} d\theta\right] \qquad \text{(D.4)}$$

The integration on θ is trivial and gives simply a factor 2π. The r integral can be evaluated by making the substitution $r^2 = u$.

$$I^2 = 2\pi\int_0^{\infty} e^{-u}\,\tfrac{1}{2}\,du = \pi \qquad \text{(D.5)}$$

Thus

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad \text{(D.6)}$$

Table I. Values of the Gauss Function*

Values of the function $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ are given for various values of t.

Each figure in the body of the table is preceded by a decimal point.

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	39894	39892	39886	39876	39862	39844	39822	39797	39767	39733
0.1	39695	39654	39608	39559	39505	39448	39387	39322	39253	39181
0.2	39104	39024	38940	38853	38762	38667	38568	38466	38361	38251
0.3	38139	38023	37903	37780	37654	37524	37391	37255	37115	36973
0.4	36827	36678	36526	36371	36213	36053	35889	35723	35553	35381
0.5	35207	35029	34849	34667	34482	34294	34105	33912	33718	33521
0.6	33322	33121	32918	32713	32506	32297	32086	31874	31659	31443
0.7	31225	31006	30785	30563	30339	30114	29887	29658	29430	29200
0.8	28969	28737	28504	28269	28034	27798	27562	27324	27086	26848
0.9	26609	26369	26129	25888	25647	25406	25164	24923	24681	24439
1.0	24197	23955	23713	23471	23230	22988	22747	22506	22265	22025
1.1	21785	21546	21307	21069	20831	20594	20357	20121	19886	19652
1.2	19419	19186	18954	18724	18494	18265	18037	17810	17585	17360
1.3	17137	16915	16694	16474	16256	16038	15822	15608	15395	15183
1.4	14973	14764	14556	14350	14146	13943	13742	13542	13344	13147
1.5	12952	12758	12566	12376	12188	12001	11816	11632	11450	11270
1.6	11092	10915	10741	10567	10396	10226	10059	09893	09728	09566
1.7	09405	09246	09089	08933	08780	08628	08478	08329	08183	08038
1.8	07895	07754	07614	07477	07341	07206	07074	06943	06814	06687
1.9	06562	06438	06316	06195	06077	05959	05844	05730	05618	05508
2.0	05399	05292	05186	05082	04980	04879	04780	04682	04586	04491
2.1	04398	04307	04217	04128	04041	03955	03871	03788	03706	03626
2.2	03547	03470	03394	03319	03246	03174	03103	03034	02965	02898
2.3	02833	02768	02705	02643	02582	02522	02463	02406	02349	02294
2.4	02239	02186	02134	02083	02033	01984	01936	01888	01842	01797
2.5	01753	01709	01667	01625	01585	01545	01506	01468	01431	01394
2.6	01358	01323	01289	01256	01223	01191	01160	01130	01100	01071
2.7	01042	01014	00987	00961	00935	00909	00885	00861	00837	00814
2.8	00792	00770	00748	00727	00707	00687	00668	00649	00631	00613
2.9	00595	00578	00562	00545	00530	00514	00499	00485	00470	00457
3.0	00443									
3.5	008727									
4.0	0001338									
4.5	0000160									
5.0	000001487									

* This table was adapted, by permission, from F. C. Kent, "Elements of Statistics," McGraw-Hill Book Company, Inc., New York, 1924.

A more complete table is "Tables of Normal Probability Functions," National Bureau of Standards, Washington, 1953.

Table II. Integrals of the Gauss Function*

Values of the integral $\frac{1}{\sqrt{2\pi}} \int_0^T e^{-t^2/2}\,dt$ are given for various values of T. To evaluate Eq. (9.22) in the text, use the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{-t^2/2}\,dt = 2\,\frac{1}{\sqrt{2\pi}} \int_0^T e^{-t^2/2}\,dt$$

A related function which is sometimes used is erf z, defined by

$$\operatorname{erf} z = \frac{1}{\sqrt{\pi}} \int_{-z}^{z} e^{-x^2}\,dx$$

The values given here are equal to ½ erf $(T/\sqrt{2})$.
Each figure in the body of the table is preceded by a decimal point.

T	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	00000	00399	00798	01197	01595	01994	02392	02790	03188	03586
0.1	03983	04380	04776	05172	05567	05962	06356	06749	07142	07535
0.2	07926	08317	08706	09095	09483	09871	10257	10642	11026	11409
0.3	11791	12172	12552	12930	13307	13683	14058	14431	14803	15173
0.4	15554	15910	16276	16640	17003	17364	17724	18082	18439	18793
0.5	19146	19497	19847	20194	20450	20884	21226	21566	21904	22240
0.6	22575	22907	23237	23565	23891	24215	24537	24857	25175	25490
0.7	25804	26115	26424	26730	27035	27337	27637	27935	28230	28524
0.8	28814	29103	29389	29673	29955	30234	30511	30785	31057	31327
0.9	31594	31859	32121	32381	32639	32894	33147	33398	33646	33891
1.0	34134	34375	34614	34850	35083	35313	35543	35769	35993	36214
1.1	36433	36650	36864	37076	37286	37493	37698	37900	38100	38298
1.2	38493	38686	38877	39065	39251	39435	39617	39796	39973	40147
1.3	40320	40490	40658	40824	40988	41149	41308	41466	41621	41774
1.4	41924	42073	42220	42364	42507	42647	42786	42922	43056	43189
1.5	43319	43448	43574	43699	43822	43943	44062	44179	44295	44408
1.6	44520	44630	44738	44845	44950	45053	45154	45254	45352	45449
1.7	45543	45637	45728	45818	45907	45994	46080	46164	46246	46327
1.8	46407	46485	46562	46638	46712	46784	46856	46926	46995	47062
1.9	47128	47193	47257	47320	47381	47441	47500	47558	47615	47670
2.0	47725	47778	47831	47882	47932	47982	48030	48077	48124	48169
2.1	48214	48257	48300	48341	48382	48422	48461	48500	48537	48574
2.2	48610	48645	48679	48713	48745	48778	48809	48840	48870	48899
2.3	48928	48956	48983	49010	49036	49061	49086	49111	49134	49158
2.4	49180	49202	49224	49245	49266	49286	49305	49324	49343	49361
2.5	49379	49396	49413	49430	49446	49461	49477	49492	49506	49520
2.6	49534	49547	49560	49573	49585	49598	49609	49621	49632	49643
2.7	49653	49664	49674	49683	49693	49702	49711	49720	49728	49736
2.8	49744	49752	49760	49767	49774	49781	49788	49795	49801	49807
2.9	49813	49819	49825	49831	49836	49841	49846	49851	49856	49861
3.0	49865									
3.5	4997674									
4.0	4999683									
4.5	4999966									
5.0	4999997133									

* This table was adapted, by permission, from F. C. Kent, "Elements of Statistics," McGraw-Hill Book Company, Inc., New York, 1924.

A more complete table is "Tables of Normal Probability Functions," National Bureau of Standards, Washington, 1953.

Table III. Maximum Deviations for Chauvenet's Criterion

For each value of N (N = number of observations) the table gives the value of d_i/σ such that the probability of occurrence of deviations *larger than* the given value is $1/2N$.

N	d_i/σ	N	d_i/σ
5	1.65	30	2.39
6	1.73	40	2.49
7	1.81	50	2.57
8	1.86	60	2.64
9	1.91	80	2.74
10	1.96	100	2.81
12	2.04	150	2.93
14	2.10	200	3.02
16	2.15	300	3.14
18	2.20	400	3.23
20	2.24	500	3.29
25	2.33	1000	3.48

Table IV. Values of χ^2*

The table gives values of χ^2 which have various probabilities of being *exceeded* by a sample taken from the given parent distribution. The number of degrees of freedom is ν. To illustrate: For a sample with 10 degrees of freedom, the probability is 0.99 that it will have $\chi^2 \geq 2.558$ and 0.001 that $\chi^2 \geq 29.588$.

ν	Probability										
	0.99	0.98	0.95	0.90	0.80	0.20	0.10	0.05	0.02	0.01	0.001
1	0.0^3157	0.0^3628	0.00393	0.0158	0.0642	1.642	2.706	3.841	5.412	6.635	10.827
2	0.0201	0.0404	0.103	0.211	0.446	3.219	4.605	5.991	7.824	9.210	13.815
3	0.115	0.185	0.352	0.584	1.005	4.642	6.251	7.815	9.837	11.341	16.268
4	0.297	0.429	0.711	1.064	1.649	5.989	7.779	9.488	11.668	13.277	18.465
5	0.554	0.752	1.145	1.610	2.343	7.289	9.236	11.070	13.388	15.086	20.517
6	0.872	1.134	1.635	2.204	3.070	8.558	10.645	12.592	15.033	16.812	22.457
7	1.239	1.564	2.167	2.833	3.822	9.803	12.017	14.067	16.622	18.475	24.322
8	1.646	2.032	2.733	3.490	4.594	11.030	13.362	15.507	18.168	20.090	26.125
9	2.088	2.532	3.325	4.168	5.380	12.242	14.684	16.919	19.679	21.666	27.877
10	2.558	3.059	3.940	4.865	6.179	13.442	15.987	18.307	21.161	23.209	29.588
11	3.053	3.609	4.575	5.578	6.989	14.631	17.275	19.675	22.618	24.725	31.264
12	3.571	4.178	5.226	6.304	7.807	15.812	18.549	21.026	24.054	26.217	32.909
13	4.107	4.765	5.892	7.042	8.634	16.985	19.812	22.362	25.472	27.688	34.528
14	4.660	5.368	6.571	7.790	9.467	18.151	21.064	23.685	26.873	29.141	36.123
15	5.229	5.985	7.261	8.547	10.307	19.311	22.307	24.996	28.259	30.578	37.697
16	5.812	6.614	7.962	9.312	11.152	20.465	23.542	26.296	29.633	32.000	39.252
17	6.408	7.255	8.672	10.085	12.002	21.615	24.769	27.587	30.995	33.409	40.790
18	7.015	7.906	9.390	10.865	12.857	22.760	25.989	28.869	32.346	34.805	42.312
19	7.633	8.567	10.117	11.651	13.716	23.900	27.204	30.144	33.687	36.191	43.820
20	8.260	9.237	10.851	12.443	14.578	25.038	28.412	31.410	35.020	37.566	45.315
21	8.897	9.915	11.591	13.240	15.445	26.171	29.615	32.671	36.343	38.932	46.797
22	9.542	10.600	12.338	14.041	16.314	27.301	30.813	33.924	37.659	40.289	48.268
23	10.196	11.293	13.091	14.848	17.187	28.429	32.007	35.172	38.968	41.638	49.728
24	10.856	11.992	13.848	15.659	18.062	29.553	33.196	36.415	40.270	42.980	51.179
25	11.524	12.697	14.611	16.473	18.940	30.675	34.382	37.652	41.566	44.314	52.620
26	12.198	13.409	15.379	17.292	19.820	31.795	35.563	38.885	42.856	45.642	54.052
27	12.879	14.125	16.151	18.114	20.703	32.912	36.741	40.113	44.140	46.963	55.476
28	13.565	14.847	16.928	18.939	21.588	34.027	37.916	41.337	45.419	48.278	56.893
29	14.256	15.574	17.708	19.768	22.475	35.139	39.087	42.557	46.693	49.588	58.302
30	14.953	16.306	18.493	20.599	23.364	36.250	40.256	43.773	47.962	50.892	59.703

* This table is reproduced in abridged form from Table IV of Fisher and Yates, "Statistical Tables for Biological, Agricultural, and Medical Research," published by Oliver & Boyd, Ltd., Edinburgh, by permission of the authors and publishers.

Table V. Correlation Coefficients*

The table gives values of the correlation coefficient r which have certain probabilities of being *exceeded* for observations of variables whose parent distributions are independent. The number of pairs of observations is N. To illustrate: for a sample of 10 pairs of observations on unrelated variables, the probability is 0.10 that it will have $r \geq 0.549$, and the probability is 0.001 that $r \geq 0.875$.

N	Probability				
	0.10	0.05	0.02	0.01	0.001
3	0.988	0.997	0.999	1.000	1.000
4	0.900	0.950	0.980	0.990	0.999
5	0.805	0.878	0.934	0.959	0.992
6	0.729	0.811	0.882	0.917	0.974
7	0.669	0.754	0.833	0.874	0.951
8	0.621	0.707	0.789	0.834	0.925
10	0.549	0.632	0.716	0.765	0.872
12	0.497	0.576	0.658	0.708	0.823
15	0.441	0.514	0.592	0.641	0.760
20	0.378	0.444	0.516	0.561	0.679
30	0.307	0.362	0.423	0.464	0.572
40	0.264	0.312	0.367	0.403	0.502
60	0.219	0.259	0.306	0.337	0.422
80	0.188	0.223	0.263	0.291	0.366
100	0.168	0.199	0.235	0.259	0.327

* This table is adapted from Table VI of Fisher and Yates, "Statistical Tables for Biological, Agricultural, and Medical Research," published by Oliver & Boyd, Ltd., Edinburgh, by permission of the authors and publishers.

BIBLIOGRAPHY

Following are a few references for readers who want to pursue further some of the topics of this book. This is not intended to be an exhaustive list of the literature, but a suggestion of a few places to look for more information. Some of the books suppose considerably more mathematical sophistication on the part of the reader than the present volume.

Dixon, W. J., and F. J. Massey, Jr.: "Introduction to Statistical Analysis," 2d ed., McGraw-Hill Book Company, Inc., New York, 1957.

Fry, Thornton C.: "Probability and Its Engineering Uses," D. Van Nostrand Company, Inc., Princeton, N.J., 1928.

Hoel, Paul G.: "Introduction to Mathematical Statistics," 2d ed., John Wiley & Sons, Inc., New York, 1954.

Jeffreys, Harold: "Theory of Probability," Oxford University Press, New York, 1948.

Lindsay, Robert B., and Henry Margenau: "Foundations of Physics," John Wiley & Sons, Inc., New York, 1936 (reprinted by Dover Publications, Inc., New York).

Munroe, Marshall E.: "The Theory of Probability," McGraw-Hill Book Company, Inc., New York, 1951.

Parratt, Lyman G.: "Probability and Experimental Errors in Science," John Wiley & Sons, Inc., New York, 1961.

Pearson, Karl: "Tables for Statisticians and Biometricians," 3d ed., Cambridge University Press, New York, 1930.

Worthing, Archie, and Joseph Geffner: "Treatment of Experimental Data," John Wiley & Sons, Inc., New York, 1943.

ANSWERS TO PROBLEMS

CHAPTER I

1. (*a*) -8.02%
   (*b*) -0.366%
2. (*a*) 0.0402%
   (*b*) 8.5×10^{-8}
3. Increases
4. 40 mph; no, 20 mph
5. 4%
6. 20.13 lb
7. (*a*) 0.5×10^{-8}
   (*b*) 0.005
9. $\frac{1}{2}n(n-1)(\delta/A)^2$
10. (*a*) 1.003
    (*b*) 1.002
    (*c*) 1.002
11. (*a*) 0.2 cm, 0.2 cm
    (*b*) 0.0019; 0.040
12. (*a*) $\theta = 0$
    (*b*) $\theta = 45°$
13. (*a*) 0.30 m/sec^2;
    0.10 m/sec^2
    (*b*) T
14. $\Delta F_x = -F \sin\theta\, \Delta\theta$
    $\Delta F_y = F \cos\theta\, \Delta\theta$
    $\Delta F_x/F_x = -\tan\theta\, \Delta\theta$
    $\Delta F_y/F_y = \cot\theta\, \Delta\theta$
15. (*a*) 0.005
    (*b*) 2×10^{-4}
    (*c*) 10^{-4}
    (*d*) 0.1
    (*e*) 0.04
16. $m = 3$, $\sigma = \sqrt{2}$, $\alpha = 6/5$
17. (*a*) 3.50
    (*b*) 4.17
    (*c*) 3.50
18. $\sigma = 0.0024$, $\alpha = 0.0018$
20. Standard deviation
21. $\alpha = \epsilon/2$, $\sigma = \epsilon/\sqrt{3}$

CHAPTER II

1. $8/76$; yes, $1/8$
2. (*a*) $5/6$
   (*b*) $(5/6)^2$
   (*c*) $(5/6)^3$
   (*d*) 0
3. 2, 3, 4, 5, 6;
   $1/4$, $1/3$, $5/18$, $1/9$, $1/36$
4. $1/64$, $3/64$, $3/32$, $5/32$, $3/16$,
   $3/16$, $5/32$, $3/32$, $3/64$, $1/64$
5. 0.706
6. $1/1326$
7. $9/29$, $153/203$, 1
8. $1/2$, $1/3$, $1/6$; yes
9. $9/50$
10. (*a*) 0.28
    (*b*) 0.010
    (*c*) 0.060
11. 0.349, 0.388, 0.263
12. Roughly 20%
13. $1/6$, $5/36$, $25/216$, $(1/6)(5/6)^n$
14. 252

15. 27,405
16. 945
17. 231,525
18. $1/270$, 725
19. $52!/39!13!$;
    no, $4 \times 52!/39!13!$
20. $1/4165$; no
22. 0.614

CHAPTER III

1. $1/64$, $3/32$, $15/64$, $5/16$, $15/64$, $3/32$, $1/64$; yes
2. $1/6$; $625/1296$, $500/1296$, $150/1296$, $20/1296$, $1/1296$, 0
3. 0.887, 0.107, 0.006
4. 4 N, 2 N, 0, 2 S, 4 S; $1/16$, $1/4$, $3/8$, $1/4$, $1/16$
5. For m blocks north, $P_m = N!/(N/2 + m/2)!(N/2 - m/2)!2^N$
6. 0; $N^{1/2}$
7. 4 N, 2 N, 0, 2 S, 4 S; $1/256$, $12/256$, $54/256$, $108/256$, $81/256$
8. For m blocks north,

$$P_m = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\left(\frac{1}{4}\right)^{(m+N)/2}\left(\frac{3}{4}\right)^{(N-m)/2}$$

9. $N/2$ blocks south;
   $(3N/4)^{1/2}$
10. (*b*) $\bar{n} = 73$; $\sigma = 27$
    (*c*) 61
11. 7.8, 2.0
12. 0.135, 0.270, 0.270,
    0.180, 0.090, 0.036,
    0.012, 0.003, 0.001; 8
13. 1.78, 1.36; 1.33
16. 0.632
17. 1.3×10^{-9}; very unlikely
18. 0.0014; coin is probably asymmetric
19. $\alpha = (2/\pi)^{1/2}\sigma$
20. 0.383
21. 0.674 σ; no; 0
22. 3, 1.22
23. (*a*) 0.175
    (*b*) 0.338
24. (*b*) b/π
    (*c*) m
    (*d*) Infinite
25. (*b*) $2b/\pi$
    (*c*) m
    (*d*) $b(4/\pi - 1)^{1/2}$
26. 0, $A/\sqrt{2}$, 4 Af, $\sqrt{2}\,\pi Af$

CHAPTER IV

1. 7.6×10^{-4}
2. 0.0114
3. $3N$
4. $\bar{x} = (\bar{x})_1/4 + 3(\bar{x})_2/4$
   $1/\sigma_m^2 = N/\sigma_1^2 + 3N/\sigma_2^2$
7. 0.22 m/sec^2

8. (a) 0.023 in., 0.017 in.
   (b) $l = 11.000 \pm 0.012$ in.,
   $w = 8.500 \pm 0.006$ in.
   (c) $A = 93.50 \pm 0.11$ in.^{2}
9. No; difference is much larger than σ
10. 0.83%; e
11. (a) 0.900 ± 0.095
    (b) $64 \pm 12°$
13. $(a/N)^{1/2}$
14. $R = \Sigma V_i^2 / \Sigma V_i I_i$
15. $x = 0.96, y = 0.015$
16. $\alpha = 30\tfrac{2}{3}°$
    $\beta = 61\tfrac{2}{3}°$
    $\gamma = 87\tfrac{2}{3}°$
    if all errors have same normal distribution
17.

$$s_0 = \frac{\begin{vmatrix} \Sigma s & \Sigma t & \Sigma t^2 \\ \Sigma st & \Sigma t^2 & \Sigma t^3 \\ \Sigma st^2 & \Sigma t^3 & \Sigma t^4 \end{vmatrix}}{\Delta}$$

$$v_0 = \frac{\begin{vmatrix} N & \Sigma s & \Sigma t^2 \\ \Sigma t & \Sigma st & \Sigma t^3 \\ \Sigma t^2 & \Sigma st^2 & \Sigma t^4 \end{vmatrix}}{\Delta}$$

$$g = \frac{\begin{vmatrix} N & \Sigma t & \Sigma s \\ \Sigma t & \Sigma t^2 & \Sigma st \\ \Sigma t^2 & \Sigma t^3 & \Sigma st^2 \end{vmatrix}}{\Delta}$$

$$\text{where } \Delta = \begin{vmatrix} N & \Sigma t & \Sigma t^2 \\ \Sigma t & \Sigma t^2 & \Sigma t^3 \\ \Sigma t^2 & \Sigma t^3 & \Sigma t^4 \end{vmatrix}$$

18. $y = 1.34x - 0.29$,
    if y errors are normal and x errors negligible
19. $\dfrac{N\Sigma I_i C_i - (\Sigma I_i)(\Sigma C_i)}{N\Sigma I_i^2 - (\Sigma I_i)^2}$
20. $3\Sigma i C_i / N(N+1)(2N+1)$

INDEX

Accuracy, 3
Answers to problems, 166–168
Approximations, and binomial theorem, 19–20
  using derivatives, 4–7
Arithmetic mean (*see* Mean)
Average (*see* Mean)
Average deviation (*see* Mean deviation)

Bibliography, 165
Binomial, expansion of, 33–34
Binomial coefficients, 32–34
  sum of, 34, 54
Binomial distribution, 48–57
  examples of, 48–53
  mean of, 54, 147–148
  normalization of, 54
  standard deviation of, 56–57, 148–150
  variance of, 56–57, 148–150
Binomial theorem, 33–34
Blunders, 3, 77

Calibration errors, 2
Cauchy distribution, 90, 96
Central-limit theorem, 64–65, 96
Chance errors, 2–3
Chauvenet's criterion, 78–80, 162
χ^2 test, 80–86, 163
  for continuous distribution, 85–86
χ^2 test, definition of, 82
  interpretation of, 83–84, 163
Combinations, 30–33
Compounding of errors (*see* Propagation of errors)
Consistency, tests for, 109–110
Correlation, 126–132
Correlation coefficient, linear, 130
  interpretation of, 130–132, 164

Data, rejection of, 76–80
Degrees of freedom, 84–85
Derivative, partial, 6–7
Deviation, 12–13
  in fit of straight line, 120
  in least-squares calculations, 112, 117, 123–125
  of mean, 93–94
Dispersion, measures of, 12–18
Distribution of random errors, 64–66
  (*See also* Binomial distribution; Gauss distribution; Parent distribution; Poisson distribution; Probability distribution; Sample distribution; Standard distribution)

Error function (*see* Gauss distribution)

Errors, calibration, 2
fractional, 6
kinds of, 2–3
propagation of, 3–8, 96–101
random, 2
systematic, 2
true, relation to deviations, 13

Factorial, 30
zero, 32–33
Fractional standard deviation, 17–18
calculations with, 99–101
Frequency, 81, 84
(*See also* Probability distribution)

Gauss distribution, 64–76
derivation of, 151–157
examples of, 67, 72, 74–76
experimental basis of, 64–66
integrals of, 158–159, 161
mean of, 70–71
mean deviation of, 73
measure of precision of, 72–73
modified, 79–80
normalization of, 69–70, 158–159
relation to binomial distribution, 157
standard deviation of, 71–72
values of, 160
variance of, 71–72
Goodness of fit, 80–86

Infinite parent distribution, 45
comparison with sample, 80–82

Least squares (*see* Method of least squares)
Least-squares sum, 105, 107, 111
and variance of mean, 110–111
Line of regression, 129

Mean, 9–12
of binomial distribution, 55–56, 147–148
definition of, 9–10
of Gauss distribution, 70–71, 73
of Poisson distribution, 60–61
of a probability distribution, 42–43
standard deviation of, 92–96
variance of, 92–96
weighted, 10–12, 108–109
standard deviation of, 18, 108–109
Mean deviation, definition of, 14–15, 17
of Gauss distribution, 73
Measure of precision of Gauss distribution, 72–73
Method of least squares, 101–126
examples of, 103, 112–119
generalizations of, 123–126
observations of unequal weight, 107–109
one unknown, 101–103
several unknowns, 115–126
straight line, 117–123

Mistakes, 3, 77
Most probable value, 104–105, 108–110
standard deviation of, 92–95, 105–109
for straight line constants, 120–121

Normal distribution (*see* Gauss distribution)
Normal equations, 118–120, 124–126
Normal error function (*see* Gauss distribution)
Normalization, binomial distribution, 54
Gauss distribution, 69–70, 158–159
Poisson distribution, 60
probability distribution, 40, 60–61, 68–69

Observation equations, 116–117, 124–126
nonlinear, 125

Parameters of distributions, 59–60, 66–73
Parent distribution, 45
comparison with sample distribution, 80–82
Partial derivative, 6–7
Per cent standard deviation, 17–18

Permutations, 29–30
Poisson distribution, 57–64
examples of, 57–58, 61–64
mean of, 60–61
normalization of, 60
relation to binomial distribution, 58–59
standard deviation of, 61
variance of, 61
Precision, 3
of the mean, 95–96
Principle, of least squares, 102, 117
(*See also* Method of least squares)
of maximum likelihood, 103–104, 107–112, 117
Probability, balls in an urn, 27–28
compound, 25–27
definition of, 23–25
for dice, 25–27
flipping pennies, 23–25
meaning of, 23–28
Probability distribution, for continuous variable, 66–71
examples of, 39–44
mean of, 42–43
meaning of, 39–42
normalization of, 40, 60–61, 68–69
standard deviation of, 44
variance of, 44
(*See also* Binomial distribution; Gauss distribution; Poisson distribution)

Probable error, 90
Problems, 18–22, 34–38, 86–90, 132–138
answers to, 166–168
Propagation of errors, 3–8, 96–101
examples of, 99–101

Radioactive decay, 57–58, 61–62
Regression, line of, 129
Rejection of data, 76–80
Residual (*see* Deviation)
Root-mean-square (rms) deviation (*see* Standard deviation)

Sample of a distribution, 45
mean from, 45–46
standard deviation from, 45–46
Sample distribution, comparison with parent distribution, 80–82
Scatter or spread (*see* Dispersion)
Standard deviation, 15–18
of binomial distribution, 56–57, 148–150
calculations with, 96–101
definition of, 15–18
fractional, 17–18
calculations with, 99–101
of a function of observables, 96–101
of Gauss distribution, 71–72
of least-squares results, 106, 108, 114, 121–123
Standard deviation, of mean, 92–96
and least-squares sum, 110–111
significance of, 93, 95–96
per cent, 17–18
of Poisson distribution, 61
of probability distribution, 44
significance of, 74–75
Summation symbol, 10, 16–17

True value, 8, 13
existence of, 8–9

Variance, of binomial distribution, 56–57, 148–150
definition of, 15
of Gauss distribution, 71–72
of least-squares results, 106, 108, 114, 121–123
of mean, 92–96
of Poisson distribution, 61
of probability distribution, 44
of variance, 106–107
of weighted mean, 18, 108–109

Weight, unequal, observations of, 9–12, 107–110
Weighted mean, 10–12, 108–109
standard deviation of, 18, 108–109
variance of, 18, 108–109

McGRAW-HILL PAPERBACKS

McGRAW-HILL PAPERBACKS

Donald H. Menzel, Howard Mumford Jones, and Lyle G. Boyd: WRITING A TECHNICAL PAPER

Clifford T. Morgan and James Deese: HOW TO STUDY

Arthur Beiser: THE WORLD OF PHYSICS