Higgs Field: Example of the Ising Model

PHYS 5163 Statistical Mechanics

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I. Abstract

The Statistical Mechanics 2011 class has included a brief overview of phase transitions and spontaneous symmetry breaking in a few classes. This paper investigates symmetry breaking and the Ising Model used in a different area of physics – the conception of the Higgs model. As in the accompanying presentation, this includes the relevant physics covered in class, followed by a discussion of ferromagnetism as an example of spontaneous symmetry breaking, using the Ising Model. Finally, this paper investigates how scientists correlated the concept of symmetry breaking when coming up with a conception of the Higgs field. Although high energy might seem a very different realm of physics than typical statistical mechanics studies, this significant overlap helped lead to some of the most fascinating exploration currently underway in high-energy physics.

II. Symmetry Breaking

Wilhelm Lenz and Ernst Ising developed the Ising model in the 1920's to describe the behavior of a material that undergoes a phase transition, a type of symmetry breaking. Symmetry breaking occurs when a system experiencing small fluctuations crosses a “critical point” (that is, one of its properties changes past a critical value, where the change in fluctuations defines the critical value). The fluctuations occurring during the critical point determine
the bifurcation of the system into a macrostate made up of those microstate fluctuations. From outside the system, where the microstate fluctuations cannot be observed, the change in the system may appear arbitrary – thus the symmetrical randomness of the system is “broken”.

The material must start with the possibility of breaking into multiple possible minima, each equally possible. One widely used illustration to understand this idea, the “Mexican hat”, shows the property capable of changing as a small ball at the top of the hat, which can roll down into multiple spots on the brim:

![Figure 1](image)

When the property/ball rolls off the top, it will continue moving till it reaches a stable point, here represented by the bottom of the hat’s brim. So the two measurable states are the symmetric state (top of hat) and the asymmetric state (a point along the bottom of the hat’s brim). The randomly chosen point around the bottom of the hat’s brim to which the ball rolls represents the lowest energy level for the given property.

Lenz and Ising developed the Ising model to study one particular phase transition in particular – that of atoms in a lattice, each with spins that can be random or aligned. Their method describes the Hamiltonian (energy) of the system due to the spins and the particle interactions:

$$H = -\sum_i \vec{m}_i \cdot \vec{B} + \frac{1}{2} \sum_{ij} U(\vec{m}_i, \vec{m}_j)$$

(1)

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http://www.nature.com/nphys/journal/v7/n1/fig_tab/nphys1874_F1.html
To solve this equation, they made some simplifications: they assumed the external magnetic field acts only in the z direction (so that the spins can only point up or down, constraining the system to a two-state representation), and that particles only interact with their four nearest neighbors in the lattice. Following the derivation from class, the equation simplifies eventually to the following:\(^2\)

\[ H = -H_0 \sum_i n_i + \frac{J}{2} \sum_{i,j}^{NN} n_i n_j \]  

\[ (2) \]

Here, the \( H_0 \) term includes the external field, and we assume that the potential represented by \( J \) is zero for all terms except the nearest neighbors. This equation can now be solved using mean field theory, as done in class, which can yield interesting physics about the system, such as the critical point – for example, for the two-state spin system, the critical temperature is the temperature at which phase transition occurs.

This specific example of symmetry breaking, the two-state spin system of atoms in a lattice, occurs in materials labeled “ferromagnetic”, as discussed in the following section.

III. Ferromagnetism

The primary example of symmetry breaking used in statistical mechanics, ferromagnetism provides a cogent summary of the relevant behavior.

At high temperatures, the atoms in ferromagnetic material have randomly aligned spins, causing zero net magnetic property of the material. When the material is cooled below its critical temperature, however (with no applied magnetic field), the material spins spontaneously change from symmetrically disordered to asymmetrically ordered:\(^3\)

\[ \]  

\[^2\] As worked out in class, April 13 2011.
At high temperatures (above the critical temperature, or “Curie Temperature”), the “exchange field” (the interactions in the material that tend to line up the magnetic field) still exists, but thermal agitation negates any orienting effect, destroying spin order (Figure 2a). Below the critical temperature, the exchange field overcomes thermal agitation and lines up the spins parallel to their neighbors, within domains, causing a net magnetic moment within the domain (Figure 2b). Complete transformation from disorder to order can occur at very low temperatures, where thermal agitation almost disappears (Figure 2c).

Considering this behavior, it becomes apparent that the state associated with the material’s lowest energy changes depending on the system’s relationship with the critical point, as seen in this qualitative plot (Figure 3).

Now one can calculate the critical temperature from a few parameters, and plot a characteristic plot of the system’s state (using $cJ = 1$):

\[ T_{\text{critical}} = \frac{1}{k_B} \ln \left( \frac{J}{\Delta} \right) \]
\[ \bar{n} = \tanh c J \bar{n} \]  
\[ T_c = \frac{cJ}{k} \]

*Figure 4*

The shape of this curve leads to what is known as the hysteresis loop, a plot of the material’s magnetism versus the external applied magnetic field (Figure 5). This represents another way to understand how the system can have nonzero magnetism for zero applied field. The in-class presentation that accompanied this paper included an interpretation of this interesting plot, with the accompanying animation (Figure 6) shown in class.

*Figure 6: QuickTime Animation*
Note the points in the animation at which the applied magnetic field changes, and the corresponding changes in the animation.

While a further detailed study of ferromagnetism eclipses the scope of this paper, the following characteristics provide at least an overview of important characteristics and their relation to the critical temperature:

- Magnetism (M), the continuous 1\textsuperscript{st} order derivative of the free energy expression, follows the relationship:

\[ M \sim (T_c - T)^\beta \]  \hspace{1cm} (5) 

- Specific heat (C), the discontinuous 2\textsuperscript{nd} order derivative of free energy, follows the expression:

\[ C \sim (T_c - T)^{-\alpha} \]  \hspace{1cm} (6) 

- Magnetic susceptibility, (X), the discontinuous 2\textsuperscript{nd} derivative of free energy, follows the expression:

\[ X \sim (T - T_c)^{-\gamma} \]  \hspace{1cm} (7) 

Clearly, certain materials act remarkably different depending on their relationship to a critical point, where the symmetrically disordered microstates can determine an asymmetrically ordered macrostate. Many areas of physics leverage this model to understand systems and material behaviors. Perhaps the most interesting topic that uses this concept, however, is the current model of the Higgs mechanism.

IV. Higgs Mechanism

Quantum field theory describes particles, fields, and waves as different representations of the same entity, and these different fields/particles/waves and the interactions between them as comprising the entirety of the universe. The “standard model” orders the way all these fields interact, as festively illustrated in Figure 7.
According to the standard model, four universal forces account for all interactions. Of these four universal forces – strong, weak, electromagnetic, and gravitational – three (all except gravitational) occur by the exchange of force-carrying particles, bosons. Different particles/fields interact by exchanging different bosons with discrete energies: the strong force interacts via gluons, the electromagnetic force interacts via gluons, and the weak force interacts via W and Z bosons. (Similarly, the gravitational force may interact via the graviton, but this has not yet been observed.)

The standard model has coalesced rapidly, as physicists have worked to understand these particle interactions. As recently as 1860, Maxwell realized that the electric and magnetic forces could be described by one force; in the 1950’s, physicists began to realize that this electromagnetic force could also have been combined with the weak force during the formation of the universe.

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However, this unified theory of an electroweak force requires massless particles. Working from this understanding, in the 1960’s, physicists Peter Higgs, Robert Brout, and Francois Englert theorized that the electromagnetic and weak forces began as unified during the big bang, and that particles had no mass just after the big bang. Then, as the temperature of the universe cooled past a critical temperature, an erratic (symmetrically disordered) field underwent a phase transition and assumed a non-zero ground state energy value everywhere in the universe. Now called the “Higgs field”, this entity causes mass by interacting with particles in the universe.\(^5\)

Physicists think that the Higgs field gives mass to any particles that interact with it by the exchange of Higgs bosons, similar to the way other forces cause phenomena such as electromagnetic fields by the exchange of different bosons. The extent to which each type of particle interacts with the Higgs field determines the amount of mass that particle has, and some particles (such as photons) do not interact at all and therefore remain massless.

According to one way to understand how Higgs bosons give mass, the Higgs field exerts a resistance, or drag, on elementary particles when they accelerate in any way, creating inertial mass from the drag. Since Einstein discovered how to equivocate inertial mass with all mass, all mass then derives from this inertial dampening from the Higgs field.

Although the quantum field theory calculations are specialized and require a great degree of definition and explanation, one can notice certain parallels in Higgs theory with this semester’s statistical mechanics studies. For example, the paper “Fate of the Standard Model with a Heavy Higgs Particle”\(^6\) includes an equation derivation for the probability density of an effective potential that leads to the state’s magnetism:

\(^5\) Higgs Field. Philosophy and History of Science, Kyoto University. [http://www.bun.kyoto-u.ac.jp/~suchii/Leib-Clk/higgs.html](http://www.bun.kyoto-u.ac.jp/~suchii/Leib-Clk/higgs.html)

\(^6\) Kuti, Julius; Lin, Lee; Shen, Yue. “Fate of the Standard Model with a Heavy Higgs Particle”. Department of Physics, U of CA at San Diego, 1988.
\[ P(\Phi_c) = \frac{e^{-\mathcal{U}(\Phi_c)}}{\int d\Phi e^{-\mathcal{U}(\Phi)}} \] (8)

In this case, the paper uses the system’s state of magnetism based on the probability density in order to develop a picture of the spontaneous symmetry breaking of the Higgs field. The related plot looks quite similar to the Mexican Hat model discussed earlier:

![Figure 8](image)

As the plot shows, a nonzero potential minimum value can occur for certain states.

V. Conclusion

As this paper has shown, the simple phenomenon of symmetry breaking has led to exciting developments in another field of physics. Physicists at high-energy colliders such as CERN are now searching for the Higgs boson, by searching over a range of possible energies/masses for evidence of the boson. The discovery or negation of the Higgs boson’s existence should either validate or force re-evaluation of the standard model, and physicists test, calculate, theorize and wait with (massive) baited breath.
Appendix: Qualifier Applications

A question about the Higgs field will probably not appear among the Statistical Mechanics qualifier questions, but a question about the concept of symmetry breaking and the Ising Model could appear. Along with understanding the Ising model and interaction simplifications, one would need to use mean field theory and the usual ensembles to solve the problem. Refer to Homework 6 for sample problems. We may additionally post another related qualifier question/solution this week in a separate document.
References

*CERN: The Standard Model.

Higgs Field. Philosophy and History of Science, Kyoto University.
   http://www.bun.kyoto-u.ac.jp/~suchii/Leib-Clk/higgs.html


Kuti, Julius; Lin, Lee; Shen, Yue. “Fate of the Standard Model with a
  Heavy Higgs Particle”. Department of Physics, U of CA at San Diego,

   http://www.nature.com/nphys/journal/v7/n1/fig_tab/nphys1874_F1.html

*The Origin of Mass in Particle Physics - Compton Lectures - The University

University of Buffalo: The Ising Model of Ferromagnetism.

Wikipedia: The Ising Model, Symmetry Breaking (primarily image use)

Gupta, Ambreesh. 60th Compton Lectures: “The Origin of Mass in Particle
   http://hep.uchicago.edu/~agupta/compton.html

* = Most interesting for entertaining extended reading